A criterion of weak compactness for operators on subspaces of Orlicz spaces

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Abstract. We give a criterion of weak compactness for the operators on the Morse-Transue space $M^\Psi$, the subspace of the Orlicz space $L^\Psi$ generated by $L^\infty$.

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1 Introduction and Notation.

In 1975, C. Niculescu established a characterization of weakly compact operators $T$ from $C(S)$, where $S$ is a compact space, into a Banach space $Z$ ([15, 16], see [4] Theorem 15.2 too): $T: C(S) \rightarrow Z$ is weakly compact if and only if there exists a Borel probability measure $\mu$ on $S$ such that, for every $\epsilon > 0$, there exists a constant $C(\epsilon) > 0$ such that:

$$
\|Tf\| \leq C(\epsilon) \|f\|_{L^1(\mu)} + \epsilon \|f\|_{\infty}, \quad \forall f \in C(S).
$$

The same kind of result was proved by H. Jarchow for $\mathbb{C}^*$-algebras in [8], and by the first author for $A(\mathbb{D})$ and $H^\infty$ (see [12]). The criterion for $H^\infty$ played a key role to give an elementary proof of the equivalence between weak compactness and compactness for composition operators on $H^\infty$.

Beside these spaces, one natural class of Banach spaces is the class of Orlicz spaces $L^\Psi$. Unfortunately, we shall see that the above criterion is in
general not true for Orlicz spaces. However, it remains true when we restrict ourselves to subspaces of the Morse-Transue space $M^\Psi$. This space is the closure of $L^\infty$ in the Orlicz space $L^\Psi$.

In this paper, we first give a characterization of the operators from a subspace of $M^\Psi$ which fix no copy of $c_0$. When the complementary function of $\Psi$ satisfies $\Delta_2$, that gives a criterion of weak compactness. If moreover $\Psi$ satisfies a growth condition, that we call $\Delta^0$, the criterion has a more usable formulation, analogous to those described above.

As in the case of $H^\infty$ (but this is far less elementary), this new version obtained for subspaces of Morse-Transue spaces (Theorem 4), combined with a study of generalized Carleson measures, may be used to prove the equivalence between weak compactness and compactness for composition operators on Hardy-Orlicz spaces (see [14]), when $\Psi$ satisfies $\Delta^0$.

In this note, we shall consider Orlicz spaces defined on a probability space $(\Omega, \mathbb{P})$, that we shall assume non purely atomic.

By an Orlicz function, we shall understand that $\Psi: [0, \infty] \to [0, \infty]$ is a non-decreasing convex function such that $\Psi(0) = 0$ and $\Psi(\infty) = \infty$. To avoid pathologies, we shall assume that we work with an Orlicz function $\Psi$ having the following additional properties: $\Psi$ is continuous at 0, strictly convex (hence strictly increasing), and such that

$$\frac{\Psi(x)}{x} \to \infty \text{ as } x \to \infty.$$  

This is essentially to exclude the case of $\Psi(x) = ax$. The Orlicz space $L^\Psi(\Omega)$ is the space of all (equivalence classes of) measurable functions $f: \Omega \to \mathbb{C}$ for which there is a constant $C > 0$ such that

$$\int_\Omega \Psi\left(\frac{|f(t)|}{C}\right) d\mathbb{P}(t) < +\infty$$

and then $\|f\|_\Psi$ (the *Luxemburg norm*) is the infimum of all possible constants $C$ such that this integral is $\leq 1$.

To every Orlicz function is associated the complementary Orlicz function $\Phi = \Psi^*: [0, \infty] \to [0, \infty]$ defined by:

$$\Phi(x) = \sup_{y \geq 0} (xy - \Psi(y)).$$

The extra assumptions on $\Psi$ ensure that $\Phi$ is itself strictly convex.
Throughout this paper, we shall assume, except explicit mention of the contrary, that the *complementary* Orlicz function satisfies the $\Delta_2$ condition ($\Phi \in \Delta_2$), i.e., for some constant $K > 0$, and some $x_0 > 0$, we have:

$$\Phi(2x) \leq K \Phi(x), \quad \forall x \geq x_0.$$  

This is usually expressed by saying that $\Psi$ satisfies the $\nabla_2$ condition ($\Psi \in \nabla_2$). This is equivalent to say that for some $\beta > 1$ and $x_0 > 0$, one has $\Psi(x) \leq \Psi(\beta x)/(2\beta)$ for $x \geq x_0$, and that implies that $\frac{\Psi(x)}{x} \rightarrow \infty$. In particular, this excludes the case $L^\Psi = L^1$.

When $\Phi$ satisfies the $\Delta_2$ condition, $L^\Psi$ is the dual space of $L^\Phi$.

We shall denote by $M^\Psi$ the closure of $L^\infty$ in $L^\Psi$. Equivalently (see [17], page 75), $M^\Psi$ is the space of (classes of) functions such that:

$$\int_\Omega \Psi \left( \frac{|f(t)|}{C} \right) dP(t) < +\infty, \quad \forall C > 0.$$  

This space is the *Morse-Transue space* associated to $\Psi$, and $(M^\Psi)^* = L^\Phi$, isometrically if $L^\Phi$ is provided with the Orlicz norm, and isomorphically if it is equipped with the Luxemburg norm (see [17], Chapter IV, Theorem 1.7, page 110).

We have $M^\Psi = L^\Psi$ if and only if $\Psi$ satisfies the $\Delta_2$ condition, and $L^\Psi$ is reflexive if and only if both $\Psi$ and $\Phi$ satisfy the $\Delta_2$ condition. When the complementary function $\Phi = \Psi^*$ of $\Psi$ satisfies it (but $\Psi$ does not satisfy this $\Delta_2$ condition, to exclude the reflexive case), we have (see [17], Chapter IV, Proposition 2.8, page 122, and Theorem 2.11, page 123):

$$(*): \quad (L^\Psi)^* = (M^\Psi)^* \oplus_1 (M^\Psi)^{\perp},$$

or, equivalently, $\quad (L^\Psi)^* = L^\Phi \oplus_1 (M^\Phi)^{\perp}$, isometrically, with the Orlicz norm on $L^\Phi$.

For all the matter about Orlicz functions and Orlicz spaces, we refer to [17], or to [10].

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Our goal in this section is the following criterion of weak compactness for operators. We begin with:

**Theorem 1** Let $\Psi$ be an arbitrary Orlicz function, and let $X$ be a subspace of the Morse-Transue space $M^\Psi$. Then an operator $T: X \to Y$ from $X$ into a Banach space $Y$ fixes no copy of $c_0$ if and only if:

$$\forall \varepsilon > 0 \, \exists C_\varepsilon > 0 : \|Tf\| \leq C_\varepsilon \int_{\Omega} \Psi\left(\varepsilon \frac{|f|}{\|f\|_\Psi}\right) dP + \varepsilon \|f\|_\Psi, \quad \forall f \in X. \quad (1)$$

Recall that saying that $T$ fixes a copy of $c_0$ means that there exists a subspace $X_0$ of $X$ isomorphic to $c_0$ such that $T$ realizes an isomorphism between $X_0$ and $T(X_0)$.

Before proving that, we shall give some consequences. First, we have:

**Corollary 2** Assume that the complementary function of $\Psi$ has $\Delta_2$ ($\Psi \in \nabla_2$). Then for every subspace $X$ of $M^\Psi$, and every operator $T: X \to Y$, $T$ is weakly compact if and only if it satisfies (1).

**Proof.** When the complementary function of $\Psi$ has $\Delta_2$, one has the decomposition ($\ast$), which means that $M^\Psi$ is $M$-ideal in its bidual (see [7], Chapter III; this result was first shown by D. Werner ([18] – see also [7], Chapter III, Example 1.4 (d), page 105 – by a different way, using the ball intersection property; note that in these references, it is moreover assumed that $\Psi$ does not satisfy the $\Delta_2$ condition, but if it satisfies it, the space $L^\Psi$ is reflexive, and so the result is obvious). But every subspace $X$ of a Banach space which is $M$-ideal of its bidual has Pelczyński’s property (V) ([5, 6]; see also [7], Chapter III, Theorem 3.4), which means that operators from $X$ are weakly compact if and only if they fix no copy of $c_0$. \[\square\]

With $\Psi$ satisfying the following growth condition, the characterization (1) takes on a more usable form.

**Definition 3** We say that the Orlicz function $\Psi$ satisfies the $\Delta^0$ condition if for some $\beta > 1$, one has:

$$\lim_{x \to +\infty} \frac{\Psi(\beta x)}{\Psi(x)} = +\infty.$$
This growth condition is a strong negation of the $\Delta_2$ condition and it implies that the complementary function $\Phi = \Psi^*$ of $\Psi$ satisfies the $\Delta_2$ condition.

**Theorem 4** Assume that $\Psi$ satisfies the $\Delta^0$ condition, and let $X$ be a subspace of $M^\Psi$. Then every linear operator $T$ mapping $X$ into some Banach space $Y$ is weakly compact if and only if for some (and then for all) $1 \leq p < \infty$:

\[
(W) \quad \forall \varepsilon > 0, \exists C_\varepsilon > 0, \quad \|T(f)\| \leq C_\varepsilon \|f\|_p + \varepsilon \|f\|_{\Psi}, \quad \forall f \in X.
\]

**Remark 1.** This theorem extends [13] Theorem II.1. As in the case of $C^*$-algebras (see [4], Notes and Remarks, Chap. 15), there are miscellaneous applications of such a characterization.

**Remark 2.** Contrary to the $\Delta_2$ condition where the constant 2 may be replaced by any constant $\beta > 1$, in this $\Delta^0$ condition, the constant $\beta$ cannot be replaced by another, as the following example shows.

**Example.** There exists an Orlicz function $\Psi$ such that:

\[
(2) \quad \lim_{x \to +\infty} \frac{\Psi(5x)}{\Psi(x)} = +\infty,
\]

but

\[
(3) \quad \liminf_{x \to +\infty} \frac{\Psi(2x)}{\Psi(x)} < +\infty.
\]

**Proof.** Let $(c_n)_n$ be an increasing sequence of positive numbers such that $\lim_{n \to \infty} \frac{c_{n+1}}{c_n} = +\infty$, take $\psi(t) = c_n$ for $t \in (4^n, 4^{n+1}]$ and $\Psi(x) = \int_0^x \psi(t) \, dt$. Then (2) is verified. On the other hand, if $x_n = 2 \cdot 4^n$, one has $\Psi(x_n) \geq c_n 4^n$, and $\Psi(2x_n) \leq c_n 4^{n+1}$, so we get (3). \qed

Before proving Theorem 4, let us note that it has the following straightforward corollary.

**Corollary 5** Let $X$ be like in Theorem 4, and assume that $\mathcal{F}$ is a family of operators from $X$ into a Banach space $Y$ with the following property: there exists a bounded sequence $(g_n)_n$ in $X$ such that $\lim_{n \to \infty} \|g_n\|_1 = 0$ and such that an operator $T \in \mathcal{F}$ is compact whenever

\[
\lim_{n \to \infty} \|Tg_n\| = 0.
\]

Then every weakly compact operator in $T \in \mathcal{F}$ is actually compact.
In the forthcoming paper [14], we prove, using a generalization of the notion of Carleson measure, that a composition operator $C_\phi: H^\psi \to H^\psi$ ($H^\psi$ is the space of analytic functions on the unit disk $\mathbb{D}$ of the complex plane whose boundary values are in $L^\psi(\partial \mathbb{D})$, and $\phi: \mathbb{D} \to \mathbb{D}$ is an analytic self-map) is compact whenever:

$$\lim_{r \to 1^-} \sup_{|\xi|=1} \Psi^{-1}\left(1/(1-r)\right)\|C_\phi(u_{\xi,r})\|_\psi = 0,$$

where:

$$u_{\xi,r}(z) = \left(\frac{1-r}{1 - \xi rz}\right)^2, \quad |z| < 1,$$

and we have:

$$\lim_{r \to 1^-} \sup_{|\xi|=1} \Psi^{-1}\left(1/(1-r)\right)\|u_{\xi,r}\|_1 = 0$$

when $C_\phi$ is weakly compact and $\Psi \in \Delta^0$.

Though the situation does not fit exactly as in Corollary 5 (not because of the space $H^\psi$, which is not a subspace of $M^\psi$: we actually work in $HM^\psi = H^\psi \cap M^\psi$ since $u_{\xi,r} \in HM^\psi$, but because of the fact that we ask a uniform limit for $|\xi| = 1$), the same ideas allow us to get, when $\Psi$ satisfies the condition $\Delta^0$, that $C_\phi$ is compact if and only if it is weakly compact.

**Proof of Theorem 4.** Assume that we have (W). We may assume that $p > 1$, since if (W) is satisfied for some $p \geq 1$, it is satisfied for all finite $p \geq 1$. Moreover, we may assume that $L^\psi \subset L^p$ since $\Psi$ satisfies condition $\Delta^0$ (since we have: $\lim_{x \to +\infty} \frac{\Psi(x)}{x^p} = +\infty$, for every $r > 0$). Then $T[(1/C_\varepsilon)B_{L^p} \cap (1/\varepsilon)B_X] \subseteq 2B_Y$. Taking the polar of these sets, we get $T^*(B_{Y^*}) \subseteq (2C_\varepsilon)B_{(L^p)^*} + (2\varepsilon)B_X^*$, for every $\varepsilon > 0$. By a well-known lemma of Grothendieck, we get, since $B_{(L^p)^*}$ is weakly compact, that $T^*(B_{Y^*})$ is relatively weakly compact, i.e. $T^*$, and hence also $T$, is weakly compact.

Conversely, assume in Theorem 4 that $T$ is weakly compact. We are going to show that (W) is satisfied with $p = 1$ (hence for all finite $p \geq 1$). Let $\varepsilon > 0$. Since the $\Delta^0$ condition implies that the complementary function of $\Psi$ satisfies the $\Delta_2$ condition, Corollary 2 implies that, when $\|f\|_\Psi = 1$:

$$\|Tf\| \leq C_{\varepsilon/2} \int_{\Omega} \Psi((\varepsilon/2)|f|) \, d\mathbb{P} + \varepsilon/2.$$

As $\Psi$ satisfies the $\Delta^0$ condition, there is some $\beta > 1$ such that $\frac{\Psi(x)}{\Psi(\beta x)} \to 0$ as $x \to \infty$; hence, with $\kappa = \varepsilon/2C_{\varepsilon/2}$, there exists some $x_\kappa > 0$ such that $\Psi(x) \leq 6$.
κΨ(βx) for x ≥ x_κ. By the convexity of Ψ, one has Ψ(x) ≤ Ψ(x_κ)x = K_κx for 0 ≤ x ≤ x_κ. Hence, for every x ≥ 0:

Ψ(x) ≤ κΨ(βx) + K_κx.

It follows that, for f ∈ X, with ∥f∥_ψ = 1:

\int_Ω Ψ((ε/2)\|f\|) \, dP ≤ κ \int_Ω Ψ(β(ε/2)\|f\|) \, dP + K_κ \frac{ε}{2} \|f\|_1 ≤ κ + K_κ \frac{ε}{2} \|f\|_1

if we have chosen ε ≤ 2/β. Hence:

∥Tf∥ ≤ C_{ε/2} \left( κ + K_κ \frac{ε}{2} \|f\|_1 \right) + \frac{ε}{2} = C_{ε/2} K_κ \|f\|_1 + \left( C_{ε/2} K_κ + \frac{ε}{2} \right) = C_{ε/2} \|f\|_1 + ε,

which is (W).

Remark. The sufficient condition is actually a general fact, which is surely well known (see [12], Theorem 1.1, for a similar result, and [4], Theorem 15.2 for C(K); see also [9], page 81), and has close connection with interpolation (see [3], Proposition 1), but we have found no reference, and so we shall state it separately without proof (the proof follows that given in [4], page 310).

**Proposition 6** Let T: X → Y be an operator between two Banach spaces. Assume that there is a Banach space Z and a weakly compact map j: X → Z such that: for every ε > 0, there exists C_ε > 0 such that

∥Tx∥ ≤ C_ε ∥jx∥_Z + ε ∥x∥_X, \quad ∀x ∈ X.

Then T is weakly compact.

Note that, by the Davis-Figiel-Johnson-Pelczyński factorization theorem, we may assume that Z is reflexive. We may also assume that j is injective, because ker j ⊆ ker T, so T induces a map T̃: X/ker j → Y with the same property as T. Indeed, if jx = 0, then ∥T̃x∥ ≤ ε ∥x∥ for every ε > 0, and hence T̃x = 0.

**Proof of Theorem 1.** Assume first that T fixes a copy of c_0. There are hence some δ > 0 and a sequence \( (f_n)_n \) in X equivalent to the canonical basis of c_0 such that \( \|f_n\|_Ψ = 1 \) and \( \|Tf_n\| ≥ δ \). In particular, there is some M > 0 such that, for every choice of ε_n = ±1:

\[ \left\| \sum_{n=1}^N ε_n f_n \right\|_Ψ ≤ M, \quad ∀N ≥ 1. \]
Let \((r_n)_n\) be a Rademacher sequence. We have, first by Khintchine’s inequality, then by Jensen’s inequality and Fubini’s Theorem:

\[
\int_\Omega \Psi \left( \frac{1}{M\sqrt{2}} \left( \sum_{n=1}^{N} |f_n|^2 \right)^{1/2} \right) \, d\mathbb{P} \leq \int_\Omega \Psi \left[ \frac{1}{M} \int_0^1 \left| \sum_{n=1}^{N} r_n(t) f_n \right| \, dt \right] \, d\mathbb{P} \\
\leq \int_\Omega \int_0^1 \Psi \left[ \frac{1}{M} \sum_{n=1}^{N} r_n(t) f_n \right] \, dt \, d\mathbb{P} \\
= \int_0^1 \int_\Omega \Psi \left[ \frac{1}{M} \sum_{n=1}^{N} r_n(t) f_n \right] \, d\mathbb{P} \, dt \leq 1.
\]

The monotone convergence Theorem gives then:

\[
\int_\Omega \Psi \left( \frac{1}{M\sqrt{2}} \left( \sum_{n=1}^{\infty} |f_n|^2 \right)^{1/2} \right) \, d\mathbb{P} \leq 1.
\]

In particular, \(\sum_{n=1}^{\infty} |f_n|^2\) is finite almost everywhere, and hence \(f_n \to 0\) almost everywhere. Since \(\Psi \left( \frac{1}{M\sqrt{2}} \left( \sum_{n=1}^{\infty} |f_n|^2 \right)^{1/2} \right) \in L^1\), by the above inequalities, Lebesgue’s dominated convergence Theorem gives:

\[
\int_\Omega \Psi \left( \frac{|f_n|}{M\sqrt{2}} \right) \, d\mathbb{P} \xrightarrow{n \to \infty} 0.
\]

But that contradicts (1) with \(\varepsilon \leq 1/M\sqrt{2}\) and \(\varepsilon < \delta\), since \(\|Tf_n\| \geq \delta\).

The converse follows from the following lemma.

**Lemma 7** Let \(X\) be a subspace of \(M^\Psi\), and let \((h_n)_n\) be a sequence in \(X\), with \(\|h_n\|_\Psi = 1\) for all \(n \geq 1\), and such that, for some \(M > 0\):

\[
\int_\Omega \Psi \left( \frac{|h_n|}{M} \right) \, d\mathbb{P} \xrightarrow{n \to \infty} 0.
\]

Then \((h_n)_n\) has a subsequence equivalent to the canonical basis of \(c_0\).

Indeed, if condition (1) is not satisfied, there exist some \(\varepsilon_0 > 0\) and functions \(h_n \in X\) with \(\|h_n\|_\Psi = 1\) such that \(\|Th_n\| \geq 2^n \int_\Omega \Psi (\varepsilon_0 |h_n|) \, d\mathbb{P} + \varepsilon_0\). That implies that \(\int_\Omega \Psi (\varepsilon_0 |h_n|) \, d\mathbb{P}\) tends to 0, so Lemma 7 ensures that \((h_n)_n\) has a subsequence, which we shall continue to denote by \((h_n)_n\), equivalent.
to the canonical basis of $c_0$. Then $(Th_n)_n$ is weakly unconditionally Cauchy. Since $\|Th_n\| \geq \varepsilon_0$, $(Th_n)_n$ has, by Bessaga-Pelczyński’s Theorem, a further subsequence equivalent to the canonical basis of $c_0$. It is then obvious that $T$ realizes an isomorphism between the spaces generated by these subsequences. □

**Proof of Lemma 7.** The proof uses the idea of the construction made in the proof of Theorem II.1 in [13], which it generalizes, but with some additional details.

By the continuity of $\Psi$, there exists $a > 0$ such that $\Psi(a) = 1$. Then, since $\Psi$ is increasing, we have, for every $g \in L^\infty$:

$$\int_{\Omega} \Psi \left( a \frac{|g|}{\|g\|_\infty} \right) \, dP \leq 1,$$

and so $\|g\|_\Psi \leq (1/a) \|g\|_\infty$.

Now, choose, for every $n \geq 1$, positive numbers $\alpha_n < a/2^{n+2}$ such that $\Psi(\alpha_n/2M) \leq 1$.

We are going to construct inductively a subsequence $(f_n)_n$ of $(h_n)_n$, a sequence of functions $g_n \in L^\infty$ and two sequences of positive numbers $\beta_n$ and $\varepsilon_n \leq \min\{1/2^{n+1}, M/2^{n+1}\}$, such that, for every $n \geq 1$:

(i) if we set $M_1 = 1$ and, for $n \geq 2$:

$$M_n = \max \left\{ 1, \Psi \left( \frac{\|g_1\|_\infty + \cdots + \|g_{n-1}\|_\infty}{2M} \right) \right\},$$

then $M_n \beta_n \leq 1/2^{n+1}$;

(ii) $\|f_n\|_\Psi = 1$;

(iii) $\|f_n - g_n\|_\Psi \leq \varepsilon_n$, with $\varepsilon_n$ such that $\beta_n \Psi(\alpha_n/2\varepsilon_n) \geq 2$;

(iv) $P(\{|g_n| > \alpha_n\}) \leq \beta_n$;

(v) $\|\hat{g}_n\|_\Psi \geq 1/2$, with $\hat{g}_n = g_n \mathbb{1}_{\{|g_n| > \alpha_n\}}$.

We shall give only the inductive step, since the starting one unfolds identically. Suppose hence that the functions $f_1, \ldots, f_{n-1}$, $g_1, \ldots, g_{n-1}$ and the numbers $\beta_1, \ldots, \beta_{n-1}$ and $\varepsilon_1, \ldots, \varepsilon_{n-1}$ have been constructed. Choose
then $\beta_n > 0$ such that $M_n \beta_n \leq 1/2^{n+1}$. Note that $M_n \geq 1$ implies that $\beta_n \leq 1/2^{n+1}$.

Since $\int_{\Omega} \Psi(|h_k|/M) \, d\mathbb{P} \to 0$ as $n \to \infty$, we can find $f_n = h_{k_n}$ such that $\|f_n\|_\Psi = 1$, and moreover:

$$\mathbb{P}(\{|f_n| > \alpha_n/2\}) \leq \frac{1}{\Psi(\alpha_n/2M)} \int_{\Omega} \Psi\left(\frac{|f_n|}{M}\right) \, d\mathbb{P} \leq \frac{\beta_n}{2}.$$  

Take now $\varepsilon_n \leq \min\{1/2^{n+1}, M/2^{n+1}\}$ such that $0 < \varepsilon_n \leq \alpha_n/2 \Psi^{-1}(2/\beta_n)$ and $g_n \in L^\infty$ such that $\|f_n - g_n\|_\Psi \leq \varepsilon_n$. Then, since

$$\mathbb{P}(\{|f_n - g_n| > \alpha_n/2\}) \Psi\left(\frac{\alpha_n}{2\varepsilon_n}\right) \leq \int_{\Omega} \Psi\left(\frac{|f_n - g_n|}{\varepsilon_n}\right) \, d\mathbb{P} \leq 1,$$

we have:

$$\mathbb{P}(\{|g_n| > \alpha_n\}) \leq \mathbb{P}(\{|f_n| > \alpha_n/2\}) + \mathbb{P}(\{|f_n - g_n| > \alpha_n/2\})$$

$$\leq \frac{\beta_n}{2} + \frac{1}{\Psi(\alpha_n/2\varepsilon_n)} \leq \beta_n.$$  

To end the construction, it remains to note that

$$\|f_n - \tilde{g}_n\|_\Psi \leq \|f_n - g_n\|_\Psi + \|\tilde{g}_n - g_n\|_\Psi \leq \varepsilon_n + \frac{1}{a} \|\tilde{g}_n - g_n\|_\infty$$

$$\leq \frac{1}{2^{n+1}} + \frac{\alpha_n}{a} \leq \frac{1}{2} \leq \frac{1}{2}$$

and so:

$$\|\tilde{g}_n\|_\Psi \geq \|f_n\|_\Psi - \|f_n - \tilde{g}_n\|_\Psi \geq 1 - \frac{1}{2} = \frac{1}{2}.$$  

This ends the inductive construction.

Consider now

$$\tilde{g} = \sum_{n=1}^{+\infty} |\tilde{g}_n|.$$  

Set $A_n = \{|g_n| > \alpha_n\}$ and, for $n \geq 1$:

$$B_n = A_n \setminus \bigcup_{j>n} A_j.$$
We have $\mathbb{P}(\limsup A_n) = 0$, because

$$\sum_{n \geq 1} \mathbb{P}(A_n) \leq \sum_{n \geq 1} \beta_n \leq \sum_{n \geq 1} \frac{1}{2^n} < +\infty.$$  

Now $\check{g}$ vanishes out of $\bigcup_{n \geq 1} B_n \cup (\limsup A_n)$ and we have:

$$\int_{B_n} \Psi\left(\frac{|\check{g}_n|}{2M}\right) d\mathbb{P} \leq \int_{\Omega} \Psi\left(\frac{|g_n|}{2M}\right) d\mathbb{P} \leq \int_{\Omega} \Psi\left(\frac{|g_n - f_n|}{2M} + \frac{|f_n|}{2M}\right) d\mathbb{P} \leq \frac{1}{2} \int_{\Omega} \Psi\left(\frac{|g_n - f_n|}{M}\right) d\mathbb{P} + \frac{1}{2} \int_{\Omega} \Psi\left(\frac{|f_n|}{M}\right) d\mathbb{P}.$$  

The first integral is less than $\varepsilon_n/M$, because $\Psi(at) \leq a \Psi(t)$ for $0 \leq a \leq 1$ and $\varepsilon_n/M \leq 1$, so that:

$$\int_{B_n} \Psi\left(\frac{|\check{g}_n|}{2M}\right) d\mathbb{P} \leq \varepsilon_n \int_{\Omega} \Psi\left(\frac{|g_n - f_n|}{M}\varepsilon_n\right) d\mathbb{P} \leq \frac{\varepsilon_n}{M} \leq \frac{1}{2^{n+1}}$$  

(since $\|f_n - g_n\| \leq \varepsilon_n$).

Since:

$$\int_{\Omega} \Psi\left(\frac{|f_n|}{M}\right) d\mathbb{P} \leq \frac{\beta_n}{2} \Psi\left(\frac{\alpha_n}{2M}\right) \leq \frac{\beta_n}{2},$$

we obtain:

$$\int_{B_n} \Psi\left(\frac{|\check{g}_n|}{2M}\right) d\mathbb{P} \leq \frac{1}{2^{n+2}} + \frac{\beta_n}{4}.$$  

Therefore, since $\mathbb{P}(B_n) \leq \mathbb{P}(A_n) \leq \beta_n$:

$$\int_{\Omega} \Psi\left(\frac{|\check{g}|}{4M}\right) d\mathbb{P} = \sum_{n=1}^{+\infty} \int_{B_n} \Psi\left(\frac{|\check{g}|}{4M}\right) d\mathbb{P}$$

$$\leq \sum_{n=1}^{+\infty} \int_{B_n} \frac{1}{2} \left[\Psi\left(\frac{\|g_1\| + \cdots + \|g_{n-1}\|}{2M}\right) + \Psi\left(\frac{|\check{g}_n|}{2M}\right)\right] d\mathbb{P}$$

by convexity of $\Psi$ and because $\check{g}_j = 0$ on $B_n$ for $j > n$

$$\leq \frac{1}{2} \sum_{n=1}^{+\infty} \left(M_n \beta_n + \frac{1}{2^{n+2}} + \frac{\beta_n}{4}\right)$$

$$\leq \frac{1}{2} \sum_{n=1}^{+\infty} \left(\frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \frac{1}{2^{n+2}}\right) \leq 1.$$  

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That proves that $\tilde{g} \in L^\Psi$, and consequently that the series $\sum_{n \geq 1} \tilde{g}_n$ is weakly unconditionally Cauchy in $L^\Psi$:

$$\sup_{n \geq 1} \sup_{\theta_k = \pm 1} \left\| \sum_{k=1}^{n} \theta_k \tilde{g}_k \right\|_\Psi \leq \sup_{n \geq 1} \left\| \tilde{g}_k \right\|_\Psi \leq \|\tilde{g}\|_\Psi \leq 4M.$$ 

Since $\|\tilde{g}_n\|_\psi \geq 1/2$, $(\tilde{g}_n)_{n \geq 1}$ has, by Bessaga-Pelczyński’s theorem, a subsequence $(\tilde{g}_{n_k})_{k \geq 1}$ which is equivalent to the canonical basis of $c_0$. The corresponding subsequence $(f_{n_k})_{k \geq 1}$ of $(f_n)_{n \geq 1}$ remains equivalent to the canonical basis of $c_0$, since

$$\sum_{n=1}^{+\infty} \|f_n - \tilde{g}_n\|_\Psi \leq \sum_{n=1}^{+\infty} \varepsilon_n + \frac{\alpha_n}{a} \leq \sum_{n=1}^{+\infty} \frac{1}{2n+1} + \frac{1}{2n+2} < 1.$$ 

That ends the proof of Lemma 7. □

### 3 Comments

**Remark 1.** Let us note that the assumption $X \subseteq M^\Psi$ cannot be relaxed in general. In fact, suppose that $X$ is a subspace of $L^\Psi$ containing $L^\infty$, and let $\xi \in (M^\psi)^* \subseteq (L^\Psi)^*$. Being of rank one, $\xi$ is trivially weakly compact. Suppose that it satisfies (W). Let $f \in X$ with norm 1, and let $\varepsilon > 0$. For $t$ large enough and $f_t = f I_{\{|f| \leq t\}}$, we have $\|f - f_t\|_2 \leq \varepsilon/C_\varepsilon$. Moreover, $f_t \in L^\infty \subseteq X$ and $\|f_t\|_\Psi \leq \|f\|_\Psi = 1$. Since $\xi$ vanishes on $L^\infty$ and $f - f_t \in X$, we get:

$$|\xi(f)| = |\xi(f - f_t)| \leq C_\varepsilon \|f - f_t\|_2 + \varepsilon \|f - f_t\|_\Psi \leq 3\varepsilon.$$ 

This implies that $\xi(f) = 0$. Since this occurs for every $\xi \in (M^\psi)^*$, we get that $X \subseteq M^\psi$ (and actually $X = M^\psi$ since $X$ contains $L^\infty$). □

In particular Theorem 4 does not hold for $X = L^\Psi$.

**Remark 2.** However, condition (W) remains true for bi-adjoint operators coming from subspaces of $M^\Psi$: if $T: X \subseteq M^\Psi \to Y$ satisfies the condition (W), then $T^{**}: X^{**} \to Y^{**}$ also satisfies it. Indeed, for every $\varepsilon > 0$, we get an equivalent norm $||| \cdot |||_e$ on $X$ by putting:

$$|||f|||_e = C_\varepsilon \|f\|_2 + \varepsilon \|f\|_\Psi.$$
Hence if \( f \in X^{\ast\ast} \), there exists a net \((f_\alpha)\) of elements in \( X \), with \( |||f_\alpha|||_\varepsilon \leq |||f|||_\varepsilon \) which converges weak-star to \( f \). Then \((Tf_\alpha)\) converges weak-star to \( T^{\ast\ast}f \), and:

\[
|||T^{\ast\ast}f||| \leq \liminf _\alpha |||Tf_\alpha||| \leq \liminf _\alpha (C_\varepsilon ||f_\alpha||_2 + \varepsilon ||f_\alpha||_\Psi)
\]

\[
= \liminf _\alpha |||f_\alpha|||_\varepsilon \leq |||f|||_\varepsilon = C_\varepsilon ||f||_2 + \varepsilon ||f||_\Psi.
\]

Hence, from Proposition 6 below, for such a \( T \), \( T^{\ast\ast} \) is weakly compact if and only if it satisfies (W). We shall use this fact in the forthcoming paper [14].

**Remark 3.** In Theorem 4, we cannot only assume that \( \Psi \notin \Delta_2 \), instead of \( \Psi \in \Delta_0 \), as the following example shows. It also shows that in Corollary 2, we cannot obtain condition (W) instead of condition (1).

**Example.** Let us define:

\[
\psi(t) = \begin{cases} 
 t & \text{for } 0 \leq t < 1, \\
 (k!)(k+2)t - k!(k+1)! & \text{for } k! \leq t \leq (k+1)!, \quad k \geq 1,
\end{cases}
\]

\( (\psi(k!)) = (k!)^2 \) for every integer \( k \geq 1 \) and \( \psi \) is linear between \( k! \) and \( (k+1)! \), and

\[
\Psi(x) = \int_0^x \psi(t) \, dt.
\]

Since \( t^2 \leq \psi(t) \) for all \( t \geq 0 \), one has \( x^3/3 \leq \Psi(x) \) for all \( x \geq 0 \). Then

\[
\Psi(2n!) \geq \int_{n!}^{2n!} \psi(t) \, dt = n!(n+2)\frac{3}{2}(n!)^2 - (n!)^2(n+1)! = (n!)^3\left(\frac{n}{2} + 2\right),
\]

whereas

\[
\Psi(n!) = \int_0^{n!} \psi(t) \, dt \leq (n!)^2 n! = (n!)^3;
\]

hence

\[
\frac{\Psi(2n!)}{\Psi(n!)} \geq \frac{n}{2} + 2,
\]

and so

\[
\limsup _{x \to +\infty} \frac{\Psi(2x)}{\Psi(x)} = +\infty,
\]

which means that \( \Psi \notin \Delta_2 \).
On the other hand, for every $\beta > 1$:

$$\Psi(n!/\beta) \geq \frac{1}{3} \left( \frac{n!}{\beta} \right)^3 = \frac{(n!)^3}{3\beta^3},$$

so

$$\frac{\Psi(n!)}{\Psi(n!/\beta)} \leq \frac{(n!)^3}{(n!)^3/3\beta^3} = 3\beta^3;$$

hence

$$\liminf_{x \to +\infty} \frac{\Psi(2^x)}{\Psi(x)} \leq 3\beta^3,$$

and $\Psi \notin \Delta^0$ (actually, this will follow too from the fact that Theorem 4 is not valid for this $\Psi$).

Moreover, the conjugate function of $\Psi$ satisfies the condition $\Delta_2$. Indeed, since $\psi$ is convex, one has $\psi(2u) \geq 2\psi(u)$ for all $u \geq 0$, and hence:

$$\Psi(2^x) = \int_0^{2^x} \psi(t) \, dt = 2 \int_0^x \psi(2u) \, du \geq 2 \int_0^x 2\psi(u) \, du = 4\Psi(x),$$

and as it was seen in the Introduction that means that $\Psi \in \nabla_2$.

Now, we have $x^3/3 \leq \Psi(x)$ for all $x \geq 0$; therefore $\|\cdot\|_3 \leq 3^{1/3} \|\cdot\|$. In particular, we have an inclusion map $j: M^\Psi \hookrightarrow L^3$, which is, of course, weakly compact. Nevertheless, assuming that $\mathbb{P}$ is diffuse, condition (W) is not verified by $j$, when $\varepsilon < 1$. Indeed, as we have seen before, one has $\Psi(n!) \leq (n!)^3$. Hence, if we choose a measurable set $A_n$ such that $\mathbb{P}(A_n) = 1/\Psi(n!)$, we have:

$$\|I_{A_n}\|_\Psi = \frac{1}{\Psi^{-1}(1/\mathbb{P}(A_n))} = \frac{1}{n!};$$

whereas:

$$\|I_{A_n}\|_3 = \mathbb{P}(A_n)^{1/3} = \frac{1}{\Psi(n!)^{1/3}} \geq \frac{1}{n!}$$

and

$$\|I_{A_n}\|_2 = \mathbb{P}(A_n)^{1/2} \leq \left[ \frac{3}{(n!)^3} \right]^{1/2} = \frac{\sqrt{3}}{(n!)^{3/2}}.$$

If condition (W) were true, we should have, for every $n \geq 1$:

$$\frac{1}{n!} \leq C_\varepsilon \frac{\sqrt{3}}{(n!)^{3/2}} + \varepsilon \frac{1}{n!},$$
that is:
\[\sqrt{n!} \leq \sqrt{3 \frac{C_{\varepsilon}}{1 - \varepsilon}},\]
which is of course impossible for \(n\) large enough. \(\square\)

**Remark 4.** In the case of the whole space \(M^\Psi\), we can give a direct proof of the necessity in Theorem 4:

Suppose that \(T : M^\Psi \to X\) is weakly compact. Then \(T^* : X^* \to L^\Phi = (M^\Psi)^*\) is weakly compact, and so the set \(K = T^*(B_{X^*})\) is relatively weakly compact.

Since \(\Phi\) satisfies the \(\Delta^0\) condition, it follows from [2] (Corollary 2.9) that \(K\) has equi-absolutely continuous norms. Hence, for every \(\varepsilon > 0\), we can find \(\delta_\varepsilon > 0\) such that:

\[m(A) \leq \delta_\varepsilon \quad \Rightarrow \quad \|g \mathbb{1}_A\|_\Phi \leq \varepsilon / 2, \quad \forall g \in T^*(B_{X^*}).\]

But (the factor 1/2 appears because we use the Luxemburg norm on the dual, and not the Orlicz norm: see [17], Proposition III.3.4):

\[
\sup_{g \in T^*(B_{X^*})} \|g \mathbb{1}_A\|_\Phi \geq \frac{1}{2} \sup_{u \in B_{X^*}} \sup_{\|f\|_\Phi \leq 1} |< f, (T^* u) \mathbb{1}_A | |
\]

\[= \frac{1}{2} \sup_{u \in B_{X^*}} \sup_{\|f\|_\Phi \leq 1} \left| \int f(T^* u) \mathbb{1}_A dm \right|
\]

\[= \frac{1}{2} \sup_{u \in B_{X^*}} \sup_{\|f\|_\Phi \leq 1} |< T(f \mathbb{1}_A), u > | = \frac{1}{2} \sup_{\|f\|_\Phi \leq 1} \|T(f \mathbb{1}_A)\|;
\]

so

\[m(A) \leq \delta_\varepsilon \quad \Rightarrow \quad \sup_{\|f\|_\Phi \leq 1} \|T(f \mathbb{1}_A)\| \leq \varepsilon.\]

Now, we have:

\[m(\{f \geq \|f\|_\Psi / 2 / \delta_\varepsilon \}) \leq \frac{\delta_\varepsilon}{\|f\|_2} \int |f| \, dm = \frac{\delta_\varepsilon}{\|f\|_2} \|f\|_1 \leq \delta_\varepsilon;\]

hence, with \(A = \{\|f\| \geq \|f\|_2 / \delta_\varepsilon \}\), we get, for \(\|f\|_\Psi \leq 1:\)

\[\|Tf\| \leq \|T(f \mathbb{1}_A)\| + \|T(f \mathbb{1}_{A^c})\| \leq \varepsilon + \|T\| \|f\|_2 \delta_\varepsilon\]

since \(|f \mathbb{1}_{A^c}| \leq \|f\|_2 / \delta_\varepsilon\) implies \(\|f \mathbb{1}_{A^c}\|_\Psi \leq \|f \mathbb{1}_{A^c}\|_\infty \leq \|f\|_2 / \delta_\varepsilon.\) \(\square\)

**Remark 5.** Conversely, E. Lavergne ([11]) recently uses our Theorem 4 to give a proof of the above quoted result of J. Alexopoulos ([2], Corollary 2.9),

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and uses it to show that, when $\Psi \in \Delta^0$, then the reflexive subspaces of $L^\Phi$ (where $\Phi$ is the conjugate of $\Psi$) are closed for the $L^1$-norm.

Another recent application of our Theorem 4 is given by I. Al Alam, which shows that if $\Psi \in \Delta^0$, then every reflexive quotient of $M^\Psi$ has a non trivial type.

References

[1] I. Al Alam, Type of reflexive quotients of the Morse-Transue space, in preparation


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