A criterion of weak compactness for operators on subspaces of Orlicz spaces

Pascal Lefèvre, Daniel Li,
Hervé Queffélec, Luis Rodríguez-Piazza

September 4, 2007

Abstact. *We give a criterion of weak compactness for the operators on the Morse-Transue space M^Ψ, the subspace of the Orlicz space L^Ψ generated by L^∞.*

Mathematics Subject Classification. Primary: 46 E 30; Secondary: 46 B 20
Key-words. Morse-Transue space; Orlicz space; weakly compact operators

1 Introduction and Notation.

In 1975, C. Niculescu established a characterization of weakly compact operators T from $\mathcal{C}(S)$, where S is a compact space, into a Banach space Z ([15, 16], see [4] Theorem 15.2 too): $T\colon \mathcal{C}(S) \to Z$ is weakly compact if and only if there exists a Borel probability measure μ on S such that, for every $\epsilon > 0$, there exists a constant $C(\epsilon) > 0$ such that:

$$\|Tf\| \leq C(\epsilon)\, \|f\|_{L^1(\mu)} + \epsilon\, \|f\|_\infty\,, \quad \forall f \in \mathcal{C}(S).$$

The same kind of result was proved by H. Jarchow for $\mathbb{C}^*$-algebras in [8], and by the first author for $A(\mathbb{D})$ and H^∞ (see [12]). The criterion for H^∞ played a key role to give an elementary proof of the equivalence between weak compactness and compactness for composition operators on H^∞.

Beside these spaces, one natural class of Banach spaces is the class of Orlicz spaces L^Ψ. Unfortunately, we shall see that the above criterion is in general not true for Orlicz spaces. However, it remains true when we restrict ourselves to subspaces of the Morse-Transue space M^Ψ. This space is the closure of L^∞ in the Orlicz space L^Ψ.

In this paper, we first give a characterization of the operators from a subspace of M^Ψ which fix no copy of c_0. When the complementary function of Ψ satisfies Δ_2, that gives a criterion of weak compactness. If moreover Ψ satisfies a growth condition, that we call Δ^0, the criterion has a more usable formulation, analogous to those described above.

As in the case of H^∞ (but this is far less elementary), this new version obtained for subspaces of Morse-Transue spaces (Theorem 4), combined with a study of generalized Carleson measures, may be used to prove the equivalence between weak compactness and compactness for composition operators on Hardy-Orlicz spaces (see [14]), when Ψ satisfies Δ^0.

In this note, we shall consider Orlicz spaces defined on a probability space $(\Omega, \mathbb{P})$, that we shall assume non purely atomic.

By an Orlicz function, we shall understand that $\Psi\colon [0,\infty] \to [0,\infty]$ is a non-decreasing convex function such that $\Psi(0) = 0$ and $\Psi(\infty) = \infty$. To avoid pathologies, we shall assume that we work with an Orlicz function Ψ having the following additional properties: Ψ is continuous at 0, strictly convex (hence *strictly* increasing), and such that

$$\frac{\Psi(x)}{x} \underset{x\to\infty}{\longrightarrow} \infty.$$

This is essentially to exclude the case of $\Psi(x) = ax$. The Orlicz space $L^\Psi(\Omega)$ is the space of all (equivalence classes of) measurable functions $f\colon \Omega \to \mathbb{C}$ for which there is a constant $C > 0$ such that

$$\int_\Omega \Psi\Big(\frac{|f(t)|}{C}\Big)\, d\mathbb{P}(t) < +\infty$$

and then $\|f\|_\Psi$ (the *Luxemburg norm*) is the infimum of all possible constants C such that this integral is ≤ 1.

To every Orlicz function is associated the complementary Orlicz function $\Phi = \Psi^*\colon [0,\infty] \to [0,\infty]$ defined by:

$$\Phi(x) = \sup_{y\geq 0} \big(xy - \Psi(y)\big).$$

The extra assumptions on Ψ ensure that Φ is itself strictly convex.

Throughout this paper, we shall assume, except explicit mention of the contrary, that the *complementary* Orlicz function satisfies the Δ_2 condition ($\Phi \in \Delta_2$), *i.e.*, for some constant $K > 0$, and some $x_0 > 0$, we have:

$$\Phi(2x) \leq K\,\Phi(x), \qquad \forall x \geq x_0.$$

This is usually expressed by saying that Ψ satisfies the ∇_2 condition ($\Psi \in \nabla_2$). This is equivalent to say that for some $\beta > 1$ and $x_0 > 0$, one has $\Psi(x) \leq \Psi(\beta x)/(2\beta)$ for $x \geq x_0$, and that implies that $\frac{\Psi(x)}{x} \underset{x\to\infty}{\longrightarrow} \infty$. In particular, this excludes the case $L^\Psi = L^1$.

When Φ satisfies the Δ_2 condition, L^Ψ is the dual space of L^Φ.

We shall denote by M^Ψ the closure of L^∞ in L^Ψ. Equivalently (see [17], page 75), M^Ψ is the space of (classes of) functions such that:

$$\int_\Omega \Psi\Big(\frac{|f(t)|}{C}\Big)\,d\mathbb{P}(t) < +\infty, \qquad \forall C > 0.$$

This space is the *Morse-Transue space* associated to Ψ, and $(M^\Psi)^* = L^\Phi$, isometrically if L^Φ is provided with the Orlicz norm, and isomorphically if it is equipped with the Luxemburg norm (see [17], Chapter IV, Theorem 1.7, page 110).

We have $M^\Psi = L^\Psi$ if and only if Ψ satisfies the Δ_2 condition, and L^Ψ is reflexive if and only if both Ψ and Φ satisfy the Δ_2 condition. When the complementary function $\Phi = \Psi^*$ of Ψ satisfies it (but Ψ does not satisfy this Δ_2 condition, to exclude the reflexive case), we have (see [17], Chapter IV, Proposition 2.8, page 122, and Theorem 2.11, page 123):

$$(*) \qquad (L^\Psi)^* = (M^\Psi)^* \oplus_1 (M^\Psi)^\perp,$$

or, equivalently, $(L^\Psi)^* = L^\Phi \oplus_1 (M^\Psi)^\perp$, isometrically, with the Orlicz norm on L^Φ.

For all the matter about Orlicz functions and Orlicz spaces, we refer to [17], or to [10].

Acknowledgement. This work was made during the stay in Lens as *Professeur invité de l'Université d'Artois*, in May–June 2005, and a visit in Lens and Lille in March 2006 of the fourth-named author. He would like to thank both Math. Departments for their kind hospitality.

2 Main result.

Our goal in this section is the following criterion of weak compactness for operators. We begin with:

Theorem 1 *Let Ψ be an arbitrary Orlicz function, and let X be a subspace of the Morse-Transue space M^Ψ. Then an operator $T\colon X \to Y$ from X into a Banach space Y fixes no copy of c_0 if and only if :*
(1)

$$\forall \varepsilon > 0 \;\; \exists C_\varepsilon > 0 : \qquad \|Tf\| \leq \bigg[C_\varepsilon \int_\Omega \Psi\Big(\varepsilon \frac{|f|}{\|f\|_\Psi}\Big)\, d\mathbb{P} + \varepsilon\bigg]\, \|f\|_\Psi, \;\; \forall f \in X.$$

Recall that saying that T fixes a copy of c_0 means that there exists a subspace X_0 of X isomorphic to c_0 such that T realizes an isomorphism between X_0 and $T(X_0)$.

Before proving that, we shall give some consequences. First, we have:

Corollary 2 *Assume that the complementary function of Ψ has Δ_2 ($\Psi \in \nabla_2$). Then for every subspace X of M^Ψ, and every operator $T\colon X \to Y$, T is weakly compact if and only if it satisfies* (1)*.*

Proof. When the complementary function of Ψ has Δ_2, one has the decomposition $(*)$, which means that M^Ψ is M-ideal in its bidual (see [7], Chapter III; this result was first shown by D. Werner ([18] – see also [7], Chapter III, Example 1.4 (d), page 105 – by a different way, using the ball intersection property; note that in these references, it is moreover assumed that Ψ does not satisfy the Δ_2 condition, but if it satisfies it, the space L^Ψ is reflexive, and so the result is obvious). But every subspace X of a Banach space which is M-ideal of its bidual has Pełczyńki's property (V) ([5, 6]; see also [7], Chapter III, Theorem 3.4), which means that operators from X are weakly compact if and only if they fix no copy of c_0. $\square$

With Ψ satisfying the following growth condition, the characterization (1) takes on a more usable form.

Definition 3 *We say that the Orlicz function Ψ satisfies the Δ^0* condition *if for some $\beta > 1$, one has:*

$$\lim_{x \to +\infty} \frac{\Psi(\beta x)}{\Psi(x)} = +\infty.$$

This growth condition is a strong negation of the Δ_2 condition and it implies that the complementary function $\Phi = \Psi^*$ of Ψ satisfies the Δ_2 condition.

Theorem 4 *Assume that Ψ satisfies the Δ^0 condition, and let X be a subspace of M^Ψ. Then every linear operator T mapping X into some Banach space Y is weakly compact if and only if for some (and then for all) $1 \leq p < \infty$:*

$$\text{(W)} \qquad \forall \varepsilon > 0,\ \exists C_\varepsilon > 0, \qquad \|T(f)\| \leq C_\varepsilon \|f\|_p + \varepsilon \, \|f\|_\Psi, \qquad \forall f \in X.$$

Remark 1. This theorem extends [13] Theorem II.1. As in the case of $\mathbb{C}^*$-algebras (see [4], Notes and Remarks, Chap. 15), there are miscellaneous applications of such a characterization.

Remark 2. Contrary to the Δ_2 condition where the constant 2 may be replaced by any constant $\beta > 1$, in this Δ^0 condition, the constant β cannot be replaced by another, as the following example shows.

Example. There exists an Orlicz function Ψ such that:

$$\lim_{x \to +\infty} \frac{\Psi(5x)}{\Psi(x)} = +\infty, \tag{2}$$

but

$$\liminf_{x \to +\infty} \frac{\Psi(2x)}{\Psi(x)} < +\infty. \tag{3}$$

Proof. Let $(c_n)_n$ be an increasing sequence of positive numbers such that $\lim\limits_{n \to \infty} \dfrac{c_{n+1}}{c_n} = +\infty$, take $\psi(t) = c_n$ for $t \in (4^n, 4^{n+1}]$ and $\Psi(x) = \int_0^x \psi(t)\, dt$. Then (2) is verified. On the other hand, if $x_n = 2 \cdot 4^n$, one has $\Psi(x_n) \geq c_n 4^n$, and $\Psi(2x_n) \leq c_n 4^{n+1}$, so we get (3). $\square$

Before proving Theorem 4, let us note that it has the following straightforward corollary.

Corollary 5 *Let X be like in Theorem 4, and assume that $\mathcal{F}$ is a family of operators from X into a Banach space Y with the following property: there exists a bounded sequence $(g_n)_n$ in X such that $\lim_{n\to\infty} \|g_n\|_1 = 0$ and such that an operator $T \in \mathcal{F}$ is compact whenever*

$$\lim_{n \to \infty} \|T g_n\| = 0.$$

Then every weakly compact operator in $T \in \mathcal{F}$ is actually compact.

In the forthcoming paper [14], we prove, using a generalization of the notion of Carleson measure, that a composition operator $C_\phi\colon H^\Psi \to H^\Psi$ (H^Ψ is the space of analytic functions on the unit disk $\mathbb{D}$ of the complex plane whose boundary values are in $L^\Psi(\partial\mathbb{D})$, and $\phi\colon \mathbb{D} \to \mathbb{D}$ is an analytic self-map) is compact whenever:

$$\lim_{r\to 1^-} \sup_{|\xi|=1} \Psi^{-1}\big(1/(1-r)\big) \|C_\phi(u_{\xi,r})\|_\Psi = 0,$$

where:

$$u_{\xi,r}(z) = \Big(\frac{1-r}{1-\bar{\xi}rz}\Big)^2, \quad |z|<1,$$

and we have:

$$\lim_{r\to 1^-} \sup_{|\xi|=1} \Psi^{-1}\big(1/(1-r)\big) \|u_{\xi,r}\|_1 = 0$$

when C_ϕ is weakly compact and $\Psi \in \Delta^0$.
Though the situation does not fit exactly as in Corollary 5 (not because of the space H^Ψ, which is not a subspace of M^Ψ: we actually work in $HM^\Psi = H^\Psi \cap M^\Psi$ since $u_{\xi,r} \in HM^\Psi$, but because of the fact that we ask a uniform limit for $|\xi| = 1$), the same ideas allow us to get, when Ψ satisfies the condition Δ^0, that C_ϕ is compact if and only if it is weakly compact.

Proof of Theorem 4. Assume that we have (W). We may assume that $p > 1$, since if (W) is satisfied for some $p \geq 1$, it is satisfied for all $q \geq p$. Moreover, we may assume that $L^\Psi \overset{j}{\hookrightarrow} L^p$ since Ψ satisfies condition Δ^0 (since we have: $\lim_{x\to+\infty} \frac{\Psi(x)}{x^r} = +\infty$, for every $r > 0$). Then $T\big[(1/C_\varepsilon)B_{L^p} \cap (1/\varepsilon)B_X\big] \subseteq 2B_Y$. Taking the polar of these sets, we get $T^*(B_{Y^*}) \subseteq (2C_\varepsilon)B_{j^*[(L^p)^*]} + (2\varepsilon)B_{X^*}$, for every $\varepsilon > 0$. By a well-known lemma of Grothendieck, we get, since $B_{j^*[(L^p)^*]}$ is weakly compact, that $T^*(B_{Y^*})$ is relatively weakly compact, *i.e.* T^*, and hence also T, is weakly compact.

Conversely, assume in Theorem 4 that T is weakly compact. We are going to show that (W) is satisfied with $p = 1$ (hence for all finite $p \geq 1$). Let $\varepsilon > 0$. Since the Δ^0 condition implies that the complementary function of Ψ satisfies the Δ_2 condition, Corollary 2 implies that, when $\|f\|_\Psi = 1$:

$$\|Tf\| \leq C_{\varepsilon/2} \int_\Omega \Psi\big((\varepsilon/2)|f|\big)\, d\mathbb{P} + \varepsilon/2.$$

As Ψ satisfies the Δ^0 condition, there is some $\beta > 1$ such that $\dfrac{\Psi(x)}{\Psi(\beta x)} \to 0$ as $x \to \infty$; hence, with $\kappa = \varepsilon/2C_{\varepsilon/2}$, there exists some $x_\kappa > 0$ such that $\Psi(x) \leq$

$\kappa\Psi(\beta x)$ for $x \geq x_\kappa$. By the convexity of Ψ, one has $\Psi(x) \leq \frac{\Psi(x_\kappa)}{x_\kappa} x =: K_\kappa x$ for $0 \leq x \leq x_\kappa$. Hence, for every $x \geq 0$:

$$\Psi(x) \leq \kappa\Psi(\beta x) + K_\kappa x.$$

It follows that, for $f \in X$, with $\|f\|_\Psi = 1$:

$$\int_\Omega \Psi\big((\varepsilon/2)|f|\big)\, d\mathbb{P} \leq \kappa \int_\Omega \Psi\big(\beta(\varepsilon/2)|f|\big)\, d\mathbb{P} + K_\kappa \frac{\varepsilon}{2}\|f\|_1 \leq \kappa + K_\kappa \frac{\varepsilon}{2}\|f\|_1$$

if we have chosen $\varepsilon \leq 2/\beta$. Hence:

$$\|Tf\| \leq C_{\varepsilon/2}\Big(\kappa + K_\kappa \frac{\varepsilon}{2}\|f\|_1\Big) + \frac{\varepsilon}{2} = C_{\varepsilon/2}\frac{\varepsilon}{2}K_\kappa\|f\|_1 + \Big(C_{\varepsilon/2}\kappa + \frac{\varepsilon}{2}\Big) = C'_\varepsilon\|f\|_1 + \varepsilon,$$

which is (W). □

Remark. The sufficient condition is actually a general fact, which is surely well known (see [12], Theorem 1.1, for a similar result, and [4], Theorem 15.2 for $\mathcal{C}(K)$; see also [9], page 81), and has close connection with interpolation (see [3], Proposition 1), but we have found no reference, and so we shall state it separately without proof (the proof follows that given in [4], page 310).

Proposition 6 *Let $T\colon X \to Y$ be an operator between two Banach spaces. Assume that there is a Banach space Z and a weakly compact map $j\colon X \to Z$ such that: for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that*

$$\|Tx\| \leq C_\varepsilon\|jx\|_Z + \varepsilon\,\|x\|_X\,, \quad \forall x \in X.$$

Then T is weakly compact.

Note that, by the Davis-Figiel-Johnson-Pełczyński factorization theorem, we may assume that Z is reflexive. We may also assume that j is injective, because $\ker j \subseteq \ker T$, so T induces a map $\tilde{T}\colon X/\ker j \to Y$ with the same property as T. Indeed, if $jx = 0$, then $\|Tx\| \leq \varepsilon\|x\|$ for every $\varepsilon > 0$, and hence $Tx = 0$.

Proof of Theorem 1. Assume first that T fixes a copy of c_0. There are hence some $\delta > 0$ and a sequence $(f_n)_n$ in X equivalent to the canonical basis of c_0 such that $\|f_n\|_\Psi = 1$ and $\|Tf_n\| \geq \delta$. In particular, there is some $M > 0$ such that, for every choice of $\varepsilon_n = \pm 1$:

$$\Big\|\sum_{n=1}^N \varepsilon_n f_n\Big\|_\Psi \leq M, \quad \forall N \geq 1.$$

Let $(r_n)_n$ be a Rademacher sequence. We have, first by Khintchine's inequality, then by Jensen's inequality and Fubini's Theorem:

$$\begin{aligned}\int_\Omega \Psi\Big(\frac{1}{M\sqrt{2}}\Big(\sum_{n=1}^N |f_n|^2\Big)^{1/2}\Big)\, d\mathbb{P} &\leq \int_\Omega \Psi\Big[\frac{1}{M}\int_0^1 \Big|\sum_{n=1}^N r_n(t) f_n\Big|\, dt\Big]\, d\mathbb{P} \\ &\leq \int_\Omega \int_0^1 \Psi\Big[\frac{1}{M}\Big|\sum_{n=1}^N r_n(t) f_n\Big|\, dt\Big]\, d\mathbb{P} \\ &= \int_0^1 \int_\Omega \Psi\Big[\frac{1}{M}\Big|\sum_{n=1}^N r_n(t) f_n\Big|\, d\mathbb{P}\Big]\, dt \leq 1.\end{aligned}$$

The monotone convergence Theorem gives then:

$$\int_\Omega \Psi\Big(\frac{1}{M\sqrt{2}}\Big(\sum_{n=1}^\infty |f_n|^2\Big)^{1/2}\Big)\, d\mathbb{P} \leq 1.$$

In particular, $\sum_{n=1}^\infty |f_n|^2$ is finite almost everywhere, and hence $f_n \to 0$ almost everywhere. Since $\Psi\Big(\frac{1}{M\sqrt{2}}\big(\sum_{n=1}^\infty |f_n|^2\big)^{1/2}\Big) \in L^1$, by the above inequalities, Lebesgue's dominated convergence Theorem gives:

$$\int_\Omega \Psi\Big(\frac{|f_n|}{M\sqrt{2}}\Big)\, d\mathbb{P} \underset{n\to\infty}{\longrightarrow} 0.$$

But that contradicts (1) with $\varepsilon \leq 1/M\sqrt{2}$ and $\varepsilon < \delta$, since $\|Tf_n\| \geq \delta$.

The converse follows from the following lemma.

Lemma 7 *Let X be a subspace of M^Ψ, and let $(h_n)_n$ be a sequence in X, with $\|h_n\|_\Psi = 1$ for all $n \geq 1$, and such that, for some $M > 0$:*

$$\int_\Omega \Psi(|h_n|/M)\, d\mathbb{P} \underset{n\to\infty}{\longrightarrow} 0.$$

Then $(h_n)_n$ has a subsequence equivalent to the canonical basis of c_0.

Indeed, if condition (1) is not satisfied, there exist some $\varepsilon_0 > 0$ and functions $h_n \in X$ with $\|h_n\|_\Psi = 1$ such that $\|Th_n\| \geq 2^n \int_\Omega \Psi(\varepsilon_0|h_n|)\, d\mathbb{P} + \varepsilon_0$. That implies that $\int_\Omega \Psi(\varepsilon_0|h_n|)\, d\mathbb{P}$ tends to 0, so Lemma 7 ensures that $(h_n)_n$ has a subsequence, which we shall continue to denote by $(h_n)_n$, equivalent to the canonical basis of c_0. Then $(Th_n)_n$ is weakly unconditionally Cauchy. Since $\|Th_n\| \geq \varepsilon_0$, $(Th_n)_n$ has, by Bessaga-Pełczyński's Theorem, a further subsequence equivalent to the canonical basis of c_0. It is then obvious that T realizes an isomorphism between the spaces generated by these subsequences. $\square$

Proof of Lemma 7. The proof uses the idea of the construction made in the proof of Theorem II.1 in [13], which it generalizes, but with some additional details.

By the continuity of Ψ, there exists $a > 0$ such that $\Psi(a) = 1$. Then, since Ψ is increasing, we have, for every $g \in L^\infty$:

$$\int_\Omega \Psi\Big(a \,\frac{|g|}{\|g\|_\infty}\Big)\, d\mathbb{P} \leq 1\,,$$

and so $\|g\|_\Psi \leq (1/a)\,\|g\|_\infty$.

Now, choose, for every $n \geq 1$, positive numbers $\alpha_n < a/2^{n+2}$ such that $\Psi(\alpha_n/2M) \leq 1$.

We are going to construct inductively a subsequence $(f_n)_n$ of $(h_n)_n$, a sequence of functions $g_n \in L^\infty$ and two sequences of positive numbers β_n and $\varepsilon_n \leq \min\{1/2^{n+1}, M/2^{n+1}\}$, such that, for every $n \geq 1$:

(i) if we set $M_1 = 1$ and, for $n \geq 2$:

$$M_n = \max\Big\{1, \Psi\Big(\frac{\|g_1\|_\infty + \cdots + \|g_{n-1}\|_\infty}{2M}\Big)\Big\}\,,$$

then $M_n\beta_n \leq 1/2^{n+1}$;

(ii) $\|f_n\|_\Psi = 1$;

(iii) $\|f_n - g_n\|_\Psi \leq \varepsilon_n$, with ε_n such that $\beta_n\Psi(\alpha_n/2\varepsilon_n) \geq 2$;

(iv) $\mathbb{P}(\{|g_n| > \alpha_n\}) \leq \beta_n$;

(v) $\|\breve{g}_n\|_\Psi \geq 1/2$, with $\breve{g}_n = g_n\, \mathbb{1}_{\{|g_n| > \alpha_n\}}$.

We shall give only the inductive step, since the starting one unfolds identically. Suppose hence that the functions $f_1, \ldots, f_{n-1}$, $g_1, \ldots, g_{n-1}$ and the numbers $\beta_1, \ldots, \beta_{n-1}$ and $\varepsilon_1, \ldots, \varepsilon_{n-1}$ have been constructed. Choose then $\beta_n > 0$ such that $M_n\beta_n \le 1/2^{n+1}$. Note that $M_n \ge 1$ implies that $\beta_n \le 1/2^{n+1}$.

Since $\int_\Omega \Psi(|h_k|/M)\, d\mathbb{P} \to 0$ as $n \to \infty$, we can find $f_n = h_{k_n}$ such that $\|f_n\|_\Psi = 1$, and moreover:

$$\mathbb{P}(\{|f_n| > \alpha_n/2\}) \le \frac{1}{\Psi(\alpha_n/2M)} \int_\Omega \Psi\Big(\frac{|f_n|}{M}\Big)\, d\mathbb{P} \le \frac{\beta_n}{2}.$$

Take now $\varepsilon_n \le \min\{1/2^{n+1}, M/2^{n+1}\}$ such that $0 < \varepsilon_n \le \alpha_n/2\Psi^{-1}(2/\beta_n)$ and $g_n \in L^\infty$ such that $\|f_n - g_n\|_\Psi \le \varepsilon_n$. Then, since

$$\mathbb{P}(\{|f_n - g_n| > \alpha_n/2\})\Psi\Big(\frac{\alpha_n}{2\varepsilon_n}\Big) \le \int_\Omega \Psi\Big(\frac{|f_n - g_n|}{\varepsilon_n}\Big)\, d\mathbb{P} \le 1,$$

we have:

$$\begin{aligned}\mathbb{P}(\{|g_n| > \alpha_n\}) &\le \mathbb{P}(\{|f_n| > \alpha_n/2\}) + \mathbb{P}(\{|f_n - g_n| > \alpha_n/2\}) \\ &\le \frac{\beta_n}{2} + \frac{1}{\Psi(\alpha_n/2\varepsilon_n)} \le \beta_n.\end{aligned}$$

To end the construction, it remains to note that

$$\begin{aligned}\|f_n - \breve{g}_n\|_\Psi &\le \|f_n - g_n\|_\Psi + \|\breve{g}_n - g_n\|_\Psi \le \varepsilon_n + \frac{1}{a}\|\breve{g}_n - g_n\|_\infty \\ &\le \frac{1}{2^{n+1}} + \frac{\alpha_n}{a} \le \frac{1}{2^n} \le \frac{1}{2}\end{aligned}$$

and so:

$$\|\breve{g}_n\|_\Psi \ge \|f_n\|_\Psi - \|f_n - \breve{g}_n\|_\Psi \ge 1 - \frac{1}{2} = \frac{1}{2}.$$

This ends the inductive construction.

Consider now

$$\breve{g} = \sum_{n=1}^{+\infty} |\breve{g}_n|\,.$$

Set $A_n = \{|g_n| > \alpha_n\}$ and, for $n \ge 1$:

$$B_n = A_n \setminus \bigcup_{j>n} A_j\,.$$

We have $\mathbb{P}\big(\limsup A_n\big) = 0$, because

$$\sum_{n\geq 1} \mathbb{P}(A_n) \leq \sum_{n\geq 1} \beta_n \leq \sum_{n\geq 1} \frac{1}{2^n} < +\infty.$$

Now $\breve{g}$ vanishes out of $\bigcup_{n\geq 1} B_n \cup \big(\limsup A_n\big)$ and we have:

$$\begin{aligned}
\int_{B_n} \Psi\Big(\frac{|\breve{g}_n|}{2M}\Big)\,d\mathbb{P} &\leq \int_\Omega \Psi\Big(\frac{|g_n|}{2M}\Big)\,d\mathbb{P} \\
&\leq \int_\Omega \Psi\Big(\frac{|g_n - f_n|}{2M} + \frac{|f_n|}{2M}\Big)\,d\mathbb{P} \\
&\leq \frac{1}{2}\int_\Omega \Psi\Big(\frac{|g_n - f_n|}{M}\Big)\,d\mathbb{P} + \frac{1}{2}\int_\Omega \Psi\Big(\frac{|f_n|}{M}\Big)\,d\mathbb{P}.
\end{aligned}$$

The first integral is less than ε_n/M, because $\Psi(at) \leq a\Psi(t)$ for $0 \leq a \leq 1$ and $\varepsilon_n/M \leq 1$, so that:

$$\int_\Omega \Psi\Big(\frac{|g_n - f_n|}{M}\Big)\,d\mathbb{P} \leq \frac{\varepsilon_n}{M}\int_\Omega \Psi\Big(\frac{|g_n - f_n|}{\varepsilon_n}\Big)\,d\mathbb{P} \leq \frac{\varepsilon_n}{M} \leq \frac{1}{2^{n+1}}$$

(since $\|f_n - g_n\|_\Psi \leq \varepsilon_n$).

Since:

$$\int_\Omega \Psi\Big(\frac{|f_n|}{M}\Big)\,d\mathbb{P} \leq \frac{\beta_n}{2}\Psi\big(\alpha_n/2M\big) \leq \beta_n/2,$$

we obtain:

$$\int_{B_n} \Psi\Big(\frac{|\breve{g}_n|}{2M}\Big)\,d\mathbb{P} \leq \frac{1}{2^{n+2}} + \frac{\beta_n}{4}\,\cdot$$

Therefore, since $\mathbb{P}(B_n) \leq \mathbb{P}(A_n) \leq \beta_n$:

$$\begin{aligned}
\int_\Omega \Psi\Big(\frac{|\breve{g}|}{4M}\Big)\,d\mathbb{P} &= \sum_{n=1}^{+\infty}\int_{B_n} \Psi\Big(\frac{|\breve{g}|}{4M}\Big)\,d\mathbb{P} \\
&\leq \sum_{n=1}^{+\infty}\int_{B_n} \frac{1}{2}\Big[\Psi\Big(\frac{\|g_1\|_\infty + \cdots + \|g_{n-1}\|_\infty}{2M}\Big) + \Psi\Big(\frac{|\breve{g}_n|}{2M}\Big)\Big]\,d\mathbb{P} \\
&\qquad \text{by convexity of } \Psi \text{ and because } \breve{g}_j = 0 \text{ on } B_n \text{ for } j > n \\
&\leq \frac{1}{2}\sum_{n=1}^{+\infty}\Big(M_n\beta_n + \frac{1}{2^{n+2}} + \frac{\beta_n}{4}\Big) \\
&\leq \frac{1}{2}\sum_{n=1}^{+\infty}\Big(\frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \frac{1}{2^{n+2}}\Big) \leq 1.
\end{aligned}$$

That proves that $\breve{g} \in L^\Psi$, and consequently that the series $\sum_{n\geq 1} \breve{g}_n$ is weakly unconditionally Cauchy in L^Ψ:

$$\sup_{n\geq 1} \sup_{\theta_k=\pm 1} \Big\| \sum_{k=1}^n \theta_k \breve{g}_k \Big\|_\Psi \leq \sup_{n\geq 1} \Big\| \sum_{k=1}^n |\breve{g}_k| \Big\|_\Psi \leq \|\breve{g}\|_\Psi \leq 4M.$$

Since $\|\breve{g}_n\|_\psi \geq 1/2$, $(\breve{g}_n)_{n\geq 1}$ has, by Bessaga-Pełczyński's theorem, a subsequence $(\breve{g}_{n_k})_{k\geq 1}$ which is equivalent to the canonical basis of c_0. The corresponding subsequence $(f_{n_k})_{k\geq 1}$ of $(f_n)_{n\geq 1}$ remains equivalent to the canonical basis of c_0, since

$$\sum_{n=1}^{+\infty} \|f_n - \breve{g}_n\|_\Psi \leq \sum_{n=1}^{+\infty} \varepsilon_n + \frac{\alpha_n}{a} \leq \sum_{n=1}^{+\infty} \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} < 1\,.$$

That ends the proof of Lemma 7. □

3 Comments

Remark 1. Let us note that the assumption $X \subseteq M^\Psi$ cannot be relaxed in general. In fact, suppose that X is a subspace of L^Ψ containing L^∞, and let $\xi \in (M^\psi)^\perp \subseteq (L^\Psi)^*$. Being of rank one, ξ is trivially weakly compact. Suppose that it satisfies (W). Let $f \in X$ with norm 1, and let $\varepsilon > 0$. For t large enough and $f_t = f\mathbb{1}_{\{|f|\leq t\}}$, we have $\|f - f_t\|_2 \leq \varepsilon/C_\varepsilon$. Moreover, $f_t \in L^\infty \subseteq X$ and $\|f_t\|_\Psi \leq \|f\|_\Psi = 1$. Since ξ vanishes on L^∞ and $f - f_t \in X$, we get:

$$|\xi(f)| = |\xi(f - f_t)| \leq C_\varepsilon \|f - f_t\|_2 + \varepsilon \|f - f_t\|_\Psi \leq 3\varepsilon.$$

This implies that $\xi(f) = 0$. Since this occurs for every $\xi \in (M^\Psi)^\perp$, we get that $X \subseteq M^\Psi$ (and actually $X = M^\Psi$ since X contains L^∞). □

In particular Theorem 4 does not hold for $X = L^\Psi$.

Remark 2. However, condition (W) remains true for bi-adjoint operators coming from subspaces of M^Ψ: if $T\colon X \subseteq M^\Psi \to Y$ satisfies the condition (W), then $T^{**}\colon X^{**} \to Y^{**}$ also satisfies it. Indeed, for every $\varepsilon > 0$, we get an equivalent norm $|||\,.\,|||_\varepsilon$ on X by putting:

$$|||f|||_\varepsilon = C_\varepsilon \|f\|_2 + \varepsilon \|f\|_\Psi.$$

Hence if $f \in X^{**}$, there exists a net $(f_\alpha)_\alpha$ of elements in X, with $|||f_\alpha|||_\varepsilon \leq |||f|||_\varepsilon$ which converges weak-star to f. Then $(Tf_\alpha)_\alpha$ converges weak-star to $T^{**}f$, and:

$$\begin{aligned}\|T^{**}f\| &\leq \liminf_\alpha \|Tf_\alpha\| \leq \liminf_\alpha (C_\varepsilon \|f_\alpha\|_2 + \varepsilon \|f_\alpha\|_\Psi) \\ &= \liminf_\alpha |||f_\alpha|||_\varepsilon \leq |||f|||_\varepsilon = C_\varepsilon \|f\|_2 + \varepsilon \|f\|_\Psi.\end{aligned}$$

Hence, from Proposition 6 below, for such a T, T^{**} is weakly compact if and only if it satisfies (W). We shall use this fact in the forthcoming paper [14].

Remark 3. In Theorem 4, we cannot only assume that $\Psi \notin \Delta_2$, instead of $\Psi \in \Delta^0$, as the following example shows. It also shows that in Corollary 2, we cannot obtain condition (W) instead of condition (1).

Example. Let us define:

$$\psi(t) = \begin{cases} t & \text{for} \quad 0 \leq t < 1, \\ (k!)(k+2)t - k!(k+1)! & \text{for} \quad k! \leq t \leq (k+1)!, \quad k \geq 1, \end{cases}$$

($\psi(k!) = (k!)^2$ for every integer $k \geq 1$ and ψ is linear between $k!$ and $(k+1)!$), and

$$\Psi(x) = \int_0^x \psi(t)\, dt.$$

Since $t^2 \leq \psi(t)$ for all $t \geq 0$, one has $x^3/3 \leq \Psi(x)$ for all $x \geq 0$. Then

$$\Psi(2.n!) \geq \int_{n!}^{2.n!} \psi(t)\, dt = n!(n+2)\frac{3}{2}(n!)^2 - (n!)^2(n+1)! = (n!)^3\Big(\frac{n}{2} + 2\Big),$$

whereas

$$\Psi(n!) = \int_0^{n!} \psi(t)\, dt \leq (n!)^2\, n! = (n!)^3\,;$$

hence

$$\frac{\Psi(2.n!)}{\Psi(n!)} \geq \frac{n}{2} + 2,$$

and so

$$\limsup_{x \to +\infty} \frac{\Psi(2x)}{\Psi(x)} = +\infty,$$

which means that $\Psi \notin \Delta_2$.

On the other hand, for every $\beta > 1$:

$$\Psi(n!/\beta) \geq \frac{1}{3}\Big(\frac{n!}{\beta}\Big)^3 = \frac{(n!)^3}{3\beta^3}\,,$$

so

$$\frac{\Psi(n!)}{\Psi(n!/\beta)} \leq \frac{(n!)^3}{(n!)^3/3\beta^3} = 3\beta^3\,;$$

hence

$$\liminf_{x\to+\infty} \frac{\Psi(2x)}{\Psi(x)} \leq 3\beta^3\,,$$

and $\Psi \notin \Delta^0$ (actually, this will follow too from the fact that Theorem 4 is not valid for this Ψ).

Moreover, the conjugate function of Ψ satisfies the condition Δ_2. Indeed, since ψ is convex, one has $\psi(2u) \geq 2\psi(u)$ for all $u \geq 0$, and hence:

$$\Psi(2x) = \int_0^{2x} \psi(t)\,dt = 2\int_0^x \psi(2u)\,du \geq 2\int_0^x 2\psi(u)\,du = 4\Psi(x),$$

and as it was seen in the Introduction that means that $\Psi \in \nabla_2$.

Now, we have $x^3/3 \leq \Psi(x)$ for all $x \geq 0$; therefore $\|\,.\,\|_3 \leq 3^{1/3}\|\,.\,\|_\Psi$. In particular, we have an inclusion map $j\colon M^\Psi \hookrightarrow L^3$, which is, of course, weakly compact. Nevertheless, assuming that $\mathbb{P}$ is diffuse, condition (W) is not verified by j, when $\varepsilon < 1$. Indeed, as we have seen before, one has $\Psi(n!) \leq (n!)^3$. Hence, if we choose a measurable set A_n such that $\mathbb{P}(A_n) = 1/\Psi(n!)$, we have:

$$\|\mathbb{1}_{A_n}\|_\Psi = \frac{1}{\Psi^{-1}\big(1/\mathbb{P}(A_n)\big)} = \frac{1}{n!}\,;$$

whereas:

$$\|\mathbb{1}_{A_n}\|_3 = \mathbb{P}(A_n)^{1/3} = \frac{1}{\Psi(n!)^{1/3}} \geq \frac{1}{n!}$$

and

$$\|\mathbb{1}_{A_n}\|_2 = \mathbb{P}(A_n)^{1/2} \leq \left[\frac{3}{(n!)^3}\right]^{1/2} = \frac{\sqrt{3}}{(n!)^{3/2}}\,.$$

If condition (W) were true, we should have, for every $n \geq 1$:

$$\frac{1}{n!} \leq C_\varepsilon\,\frac{\sqrt{3}}{(n!)^{3/2}} + \varepsilon\,\frac{1}{n!}\,,$$

that is:

$$\sqrt{n!} \leq \sqrt{3}\,\frac{C_\varepsilon}{1-\varepsilon}\,,$$

which is of course impossible for n large enough. $\square$

Remark 4. In the case of the whole space M^Ψ, we can give a direct proof of the necessity in Theorem 4:

Suppose that $T\colon M^\Psi \to X$ is weakly compact. Then $T^*\colon X^* \to L^\Phi = (M^\Psi)^*$ is weakly compact, and so the set $K = T^*(B_{X^*})$ is relatively weakly compact.

Since Φ satisfies the Δ^0 condition, it follows from [2] (Corollary 2.9) that K has equi-absolutely continuous norms. Hence, for every $\varepsilon > 0$, we can find $\delta_\varepsilon > 0$ such that:

$$m(A) \leq \delta_\varepsilon \quad \Rightarrow \quad \|g\mathbb{1}_A\|_\Phi \leq \varepsilon/2\,, \quad \forall g \in T^*(B_{X^*}).$$

But (the factor $1/2$ appears because we use the Luxemburg norm on the dual, and not the Orlicz norm: see [17], Proposition III.3.4):

$$\begin{aligned}
\sup_{g\in T^*(B_{X^*})} \|g\mathbb{1}_A\|_\Phi &\geq \frac{1}{2}\sup_{u\in B_{X^*}}\sup_{\|f\|_\Psi\leq 1} |< f, (T^*u)\mathbb{1}_A >| \\
&= \frac{1}{2}\sup_{u\in B_{X^*}}\sup_{\|f\|_\Psi\leq 1} \Big| \int f(T^*u)\mathbb{1}_A\,dm\Big| \\
&= \frac{1}{2}\sup_{u\in B_{X^*}}\sup_{\|f\|_\Psi\leq 1} |< T(f\mathbb{1}_A), u >| = \frac{1}{2}\sup_{\|f\|_\Psi\leq 1}\|T(f\mathbb{1}_A)\|;
\end{aligned}$$

so

$$m(A) \leq \delta_\varepsilon \quad \Rightarrow \quad \sup_{\|f\|_\Psi\leq 1}\|T(f\mathbb{1}_A)\| \leq \varepsilon.$$

Now, we have:

$$m(|f| \geq \|f\|_2/\delta_\varepsilon) \leq \frac{\delta_\varepsilon}{\|f\|_2}\int |f|\,dm = \frac{\delta_\varepsilon}{\|f\|_2}\|f\|_1 \leq \delta_\varepsilon;$$

hence, with $A = \{|f| \geq \|f\|_2/\delta_\varepsilon\}$, we get, for $\|f\|_\Psi \leq 1$:

$$\|Tf\| \leq \|T(f\mathbb{1}_A)\| + \|T(f\mathbb{1}_{A^c})\| \leq \varepsilon + \|T\|\frac{\|f\|_2}{\delta_\varepsilon}$$

since $|f\mathbb{1}_{A^c}| \leq \|f\|_2/\delta_\varepsilon$ implies $\|f\mathbb{1}_{A^c}\|_\Psi \leq \|f\mathbb{1}_{A^c}\|_\infty \leq \|f\|_2/\delta_\varepsilon$. $\square$

Remark 5. Conversely, E. Lavergne ([11]) recently uses our Theorem 4 to give a proof of the above quoted result of J. Alexopoulos ([2], Corollary 2.9), and uses it to show that, when $\Psi \in \Delta^0$, then the reflexive subspaces of L^Φ (where Φ is the conjugate of Ψ) are closed for the L^1-norm.

Another recent application of our Theorem 4 is given by I. Al Alam, which shows that if $\Psi \in \Delta^0$, then every reflexive quotient of M^Ψ has a non trivial type.

References

[1] I. Al Alam, Type of reflexive quotients of the Morse-Transue space, *in preparation*

[2] J. Alexopoulos, De la Vallée–Poussin's theorem and weakly compact sets in Orlicz spaces, Quaestiones Math. 17 (1994), 231–248.

[3] B. Beauzamy, Propriétés géométriques des espaces d'interpolation, Séminaire Maurey-Schwartz 1974–1975: Espaces L_p, applications radonifiantes et géométrie des espaces de Banach, Exp. No. XIV, 17 pp., Centre Math., École Polytech., Paris, 1975.

[4] J. Diestel, H. Jarchow and A. Tonge, Absolutely summing operators, Cambridge studies in advanced mathematics 43, Cambridge University Press (1995).

[5] G. Godefroy et P. Saab, Quelques espaces de Banach ayant les propriétés (V) ou (V^*) de A. Pełczyński, C.R. Acad. Sci. Paris, Série A 303 (1986), 503–506.

[6] G. Godefroy and P. Saab, Weakly unconditionally convergent series in M-ideals, Math. Scand. 64 (1989), 307–318.

[7] P. Harmand, D. Werner and W. Werner, M-ideals in Banach Spaces and Banach Algebras, Lecture Notes in Math. 1547, Springer (1993).

[8] H. Jarchow, On Weakly Compact Operators on $\mathbf{C}^*$-Algebras, Math. Ann. 273 (1986), 341–343.

[9] H. Jarchow and U. Matter, On weakly compact operators on $\mathcal{C}(K)$-spaces, 80–88, Proceedings of the Missouri Conference held in Columbia, USA, June 24–29 1984, N. Kalton and E. Saab, Eds., Lecture Notes in Math. 1166 (1985).

[10] M. A. Krasnosel'skiĭ and Ya. B. Rutickiĭ, Convex functions and Orlicz spaces (translation), P. Noordhoff Ltd., Groningen (1961).

[11] E. Lavergne, Reflexive subspaces of some Orlicz spaces, *preprint.*

[12] P. Lefèvre, Some characterizations of weakly compact operators on H^∞ and on the disk algebra. Application to composition operators, Journal of Operator Theory, 54 (2) (2005), 229–238.

[13] P. Lefèvre, D. Li, H. Queffélec and L. Rodríguez-Piazza, Some translation-invariant Banach function spaces which contain c_0, Studia Math. 163 (2) (2004), 137–155.

[14] P. Lefèvre, D. Li, H. Queffélec and L. Rodríguez-Piazza, Composition operators on Hardy-Orlicz spaces, *preprint.*

[15] C. P. Niculescu, Absolute continuity and weak compactness, Bull. Amer. Math. Soc. 81 (1975), 1064–1066.

[16] C. P. Niculescu, Absolute continuity in Banach space theory, Rev. Roum. Math. Pures Appl. 24 (1979), 413–422.

[17] M. M. Rao and Z. D. Ren, Theory of Orlicz spaces, Pure and Applied Mathematics 146, Marcel Dekker, Inc. (1991).

[18] D. Werner, New classes of Banach spaces which are M-ideals in their biduals, Math. Proc. Cambridge Philos. Soc. 111 (2) (1992), 337–354.

P. Lefèvre and D. Li, Université d'Artois, Laboratoire de Mathématiques de Lens EA 2462, Fédération CNRS Nord-Pas-de-Calais FR 2956, Faculté des Sciences Jean Perrin, Rue Jean Souvraz, S.P. 18,
62 307 LENS Cedex, FRANCE
pascal.lefevre@euler.univ-artois.fr – daniel.li@euler.univ-artois.fr

H. Queffélec, Université des Sciences et Techniques de Lille, Laboratoire Paul Painlevé U.M.R. CNRS 8524, U.F.R. de Mathématiques,
59 655 VILLENEUVE D'ASCQ Cedex, FRANCE
queff@math.univ-lille1.fr

Luis Rodríguez-Piazza, Universidad de Sevilla, Facultad de Matematicas, Dpto de Análisis Matemático, Apartado de Correos 1160,
41 080 SEVILLA, SPAIN
piazza@us.es