Generalized essential norm of weighted composition operators on some uniform algebras of analytic functions.

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Abstract

We compute the essential norm of a composition operator relatively to the class of Dunford-Pettis opertors or weakly compact operators, on some uniform algebras of analytic functions. Even in the frame of H^∞ (resp. the disk algebra), this is new, as well for the polydisk algebras and the polyball algebras. This is a consequence of a general study of weighted composition operators.

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0 Introduction

The aim of this paper is to investigate the potential default of complete continuity or weak compactness of a composition operators on uniform algebras of analytic functions. This includes the case of the space H^∞ of bounded analytic functions on the open unit disk $\mathbb{D}$ of the complex plane, the case of the disk algebra $A(\mathbb{D})$, and more generally the polydisk and polyball algebras.

The composition operators were investigated so far in many ways. They are very often investigated on H^p spaces ($1 < p < \infty$). But on such spaces, their weak compactness and complete continuity are trivial problems (because of reflexivity). The monographs [6] and [17] are very good surveys on this topic. Another very interesting survey is the paper [13].

The characterization of the weak compactness for composition operators on the classical spaces H^∞ and $A(\mathbb{D})$ is not new : this is due to Aron, Galindo and Lindström [1] (see Ülger [18] too). The author recovered this result in [16] as a consequence of a characterization of (general) weakly compact operators on H^∞. The characterization of complete continuity of composition operators was settled in [12] (see [11] too).

On the other hand, weak compactness and complete continuity of the multiplication operator were considered in [14], but for spaces $C(K)$, L^1 or H^1.

We are going to use very elementary technics to estimate the essential norm relatively to Dunford-Pettis operators and weakly compact operators. This generalizes the result of Zheng [19] on the (classical) essential norm of a composition operator on H^∞. Moreover, the results cited above ([1],[12]) are recovered in a very simple way. We generalize too all these results, mixing classical composition operators with multiplication operators, i.e. studying weighted composition operators. The compact case was already considered on the disk algebra by Kamowitz in [15]. In that direction too, the results are generalized. Weighted composition operators were also studied by Contreras and Hernández-Díaz in [5] but on spaces H^p with $1 \leq p < \infty$.

In the first part, we treat the classical case of one variable analytic functions. In the second part, we adapt the method to treat the case of general analytic functions on the open unit ball of a complex Banach space. Even for the Calkin algebra, as far as we know, this is new.

In the sequel, we use different classical classes of operators ideals. We refer to [10] to know more on the subject. Recall that an operator $T : X \to Y$ is a weakly compact operator if T maps the unit ball of X into a relatively weakly compact set. Now, we precise the definitions of two properties sometimes shared by Banach spaces.

Definition 0.1 *Let X be a Banach space. X has the Dunford-Pettis property if for every weakly null sequence (x_n) in X and every weakly null sequence (x_n^*) in X^*, then $x_n^*(x_n)$ tends to zero.*

Equivalently, for every Banach space Y and every operator $T : X \to Y$ which is weakly compact, T maps a weakly Cauchy sequence in X into a norm Cauchy sequence.

A good survey on the subject (until the early eighties) is the paper of Diestel [8].

Definition 0.2 *Let X be a Banach space. X has the property (V) of Pełczyński if, for every non relatively weakly compact bounded set $K \subset X^*$, there exists a weakly unconditionaly series $\sum x_n$ in X such that $\inf_n \sup\{|k(x_n)|;\ k \in K\} > 0$.*

Equivalently, for every Banach space Y and every operator $T : X \to Y$ which is not weakly compact, there exists a subspace X_o of X isomorphic to c_o such that $T_{|X_o}$ is an isomorphic embedding.

We are going to extend results known for the Calkin algebra, to some other ideals of operators. Let us precise the terminology.

Definition 0.3 *Let X, Y be Banach spaces and $\mathcal{I}$ be a closed subspace of the space $B(X,Y)$ of bounded operators from X to Y. The essential norm (relatively to $\mathcal{I}$) of $T \in B(X,Y)$ is the distance from T to $\mathcal{I}$:*

$$\|T\|_{e,\mathcal{I}} = \inf\{\|T+S\|;\ S \in \mathcal{I}\}.$$

This is the canonical norm on the quotient space $B(X,Y)/\mathcal{I}$.
If moreover $\mathcal{I}$ is an ideal of the space $B(X)$ then $B(X)/\mathcal{I}$ is a Banach algebra.

The classical case corresponds to the case of compact operators $\mathcal{I} = \mathcal{K}(X,Y)$ (in this case, the preceding quotient space is the Calkin algebra). In the sequel, we get interested in the case of weakly compact operators: $\mathcal{I} = \mathcal{W}(X,Y)$; and in the case of completely continuous operators (sometimes also named Dunford-Pettis operators): $\mathcal{I} = \mathcal{DP}(X,Y)$. Recall that a completely continuous operator maps a weakly Cauchy sequence into a norm Cauchy sequence. Compact operators are both weakly compact and completely continuous.

Note that $X = H^\infty$ (resp. the disk algebra, the polydisk algebra and the polyball algebra) has the Dunford-Pettis property (see [4], [2] and [3]) and this can be reformulated as $\mathcal{W}(X,Y) \subset \mathcal{DP}(X,Y)$, for every Banach space Y. On the other hand, $X = H^\infty$ (resp. the disk algebra, the polydisk algebra and the polyball algebra) has the Pełczyński property and this implies that $\mathcal{DP}(X,Y) \subset \mathcal{W}(X,Y)$ (see [7], [2] and [3]). So actually, in this framework, the two notions of weak compactness and complete continuity coincide. Nevertheless, we are going to give self-contained proofs, hence we do not require the (non-trivial) results cited above. That's why we are able to treat the case of more general spaces, where such properties are not known.

We introduce the following operator: given an analytic bounded function u on the unit ball B of the Banach space E and an analytic function φ from B to B, we shall estimate in the paper the (generalized) essential norm of the weighted composition operator $T_{u,\varphi}$:

$$T_{u,\varphi}(f) = u.f \circ \varphi \qquad \text{where } f \text{ is analytic on } B.$$

Of course, when $u = \mathbb{1}$, this operator is a classical composition operator and is simply denoted by C_φ. When $\varphi = Id_B$, this operator is the multiplication operator by u.

Observe that $T_{u,\varphi}$ is always bounded from the space of bounded analytic functions on B to the space of analytic bounded functions on B. Moreover, we have $\|T_{u,\varphi}\| = \|u\|_\infty$, where $\|u\|_\infty = \sup\{|u(z)|;\ z \in B\}$.

The following quantity plays a crucial role in the estimate of the essential norm: we define

$$n_\varphi(u) = \overline{\lim_{\substack{\|\varphi(z)\| \to 1^- \\ z \in B}}} |u(z)| = \lim_{r \to 1^-} \sup\left\{|u(z)|;\ z \in B \text{ and } \|\varphi(z)\| \geq r\right\}$$

which defines a finite number since u is bounded.

When $\|\varphi\|_\infty < 1$, i.e. $\overline{\varphi(B)} \cap \partial B = \emptyset$, then $n_\varphi(u) = 0$ (i.e. the supremum over the empty set is taken as 0).

Following the spirit of this area of mathematics, we consider analytic functions: we shall assume in the paper that u is analytic, nevertheless all the results can be easily adapted under the only assumption of continuity. Under such an assumption, the operators are not defined anymore into H^∞, but the space of continuous bounded functions on B. It is actually easier to study these operators, removing the analycity of u.

1 The one variable case.

In that section, we are going to compute the essential norm relatively both to the completely continuous operators and to the weakly compact operators, on H^∞ and $A(\mathbb{D})$. We first establish the following characterization, which is a generalization of [15], Th.1.2.

With the previous notations.

Theorem 1.1 *The following assertions are equivalent*

1) $T_{u,\varphi} : A(\mathbb{D}) \to H^\infty$ *is completely continuous.*

2) $T_{u,\varphi} : A(\mathbb{D}) \to H^\infty$ *is weakly compact.*

3) $n_\varphi(u) = 0$.

4) $T_{u,\varphi} : H^\infty \to H^\infty$ *is compact.*

In the previous statement, the third assertion clearly implies that we have $u^*_{|\varphi*^{-1}(\mathbb{T})} = 0$ (a.e.), where, as usual, we denote by φ^*, resp. u^*, the boundary values of φ, resp. u (defined almost everywhere on $\mathbb{T}$ by radial limit).

An important point of the proof is that we are going to avoid the requirement to the Dunford-Pettis property, as this will be done in the general case (see the second section below).

Proof: Obviously 4 implies 1 and 2.

$1 \Rightarrow 3$. Assume that $\|\varphi\|_\infty = 1$ and $n_\varphi(u) > \varepsilon_0 > 0$.

Choose any sequence $(z_j)_j$ in the unit disk $\mathbb{D}$ such that $|\varphi(z_j)|$ converges to 1 and $|u(z_j)| \geq \varepsilon_0$. Extracting a subsequence if necessary, we may suppose that $\varphi(z_j)$ converges to some a, belonging to the torus. Now, we consider the sequence of functions $f_n(z) = 2^{-n}(\bar{a}z+1)^n$, which lies in the unit ball of the disk algebra. This is clearly a weakly-Cauchy sequence: for every $z \in \overline{\mathbb{D}} \setminus \{a\}$, $f_n(z) \to 0$; and $f_n(a) = 1$.

Since the operator $T_{u,\varphi}$ is completely continuous, the sequence $\big(u.f_n \circ \varphi\big)_{n\in\mathbb{N}}$ is norm-Cauchy, hence converges to some $\sigma \in H^\infty$. But for every fixed $z \in \mathbb{D}$, $u(z).f_n \circ \varphi(z)$ converges both to 0 and $\sigma(z)$, so that $\sigma = 0$.

Fixing $\varepsilon > 0$, this gives n_0 such that $\sup_{z\in\mathbb{D}} \big|u(z)f_{n_0} \circ \varphi(z)\big| \leq \varepsilon \cdot$

Choosing $z = z_{j_0}$ with j_0 large enough to have $\big|f_{n_0} \circ \varphi(z_{j_0})\big| \geq 1 - \varepsilon$, we have:

$$\varepsilon \geq \big|u\big(z_{j_0}\big)\big|(1-\varepsilon) \geq (1-\varepsilon)\varepsilon_0.$$

As ε is arbitrary, this gives a contradiction.

$2 \Rightarrow 3$. Assume that $\|\varphi\|_\infty = 1$ and $n_\varphi(u) > \varepsilon_0 > 0$. The idea is very close to the previous argument. Choose any sequence $z_j \in \mathbb{D}$ such that $|\varphi(z_j)|$ converges to 1 and $|u(z_j)| \geq \varepsilon_0$. We may assume that $\varphi(z_j)$ converges to some $a \in \mathbb{T}$. Now, we consider the same sequence of functions $f_n(z) = 2^{-n}(\bar{a}z+1)^n$.

Since the operator $T_{u,\varphi}$ is a weakly compact operator, there exists a sequence on integers (n_k) such that $\big(u.f_{n_k}\circ\varphi\big)_{k\in\mathbb{N}}$ is weakly convergent to some $\sigma\in H^\infty$. Testing the weak convergence on the Dirac $\delta_z\in(H^\infty)^*$, for every fixed $z\in\mathbb{D}$, we obtain that $\sigma=0$.

By the Mazur Theorem, there exists a convex combination of these functions which is norm convergent to 0:

$$\sum_{k\in I_m} c_k\, u.f_{n_k}\circ\varphi \longrightarrow 0$$

where $c_k\geq 0$ and $\displaystyle\sum_{k\in I_m} c_k=1$.

Now, fixing $\varepsilon\in\big(0,\varepsilon_0/2\big)$, we have for a suitable m_0 and every j

$$\begin{aligned}\varepsilon_0\Big|\textstyle\sum_{k\in I_{m_0}} c_k.f_{n_k}\big(\varphi(z_j)\big)\Big| &\leq \Big|\textstyle\sum_{k\in I_{m_0}} c_k\, u(z_j).f_{n_k}\big(\varphi(z_j)\big)\Big|\\ &\leq \sup_{z\in\mathbb{D}}\Big|\sum_{k\in I_{m_0}} c_k\, u(z).f_{n_k}\big(\varphi(z)\big)\Big|\\ &\leq \varepsilon.\end{aligned}$$

Letting j tends to infinity, we have $f_{n_k}\big(\varphi(z_j)\big)\to f_{n_k}(a)=1$ for each k so that

$$\varepsilon_0=\varepsilon_0\Big|\sum_{k\in I_{m_0}} c_k\Big|\leq\varepsilon.$$

This gives a contradiction.

$3\Rightarrow 4$. Point out that $T_{u,\varphi}=M_u\circ C_\varphi$.

If $\|\varphi\|_\infty<1$ then C_φ is compact (see the remark below).

If $\|\varphi\|_\infty=1$ and $\displaystyle\lim_{\substack{\|\varphi(z)\|\to1^-\\ z\in\mathbb{D}}} u(z)=0$ then $T_{u,\varphi}$ is compact. Indeed, given a sequence in the unit ball of H^∞, we can extract a subsequence $(f_n)_n$ converging on every compact subsets of $\mathbb{D}$. Given $\varepsilon>0$, we choose a compact disk $K\subset\mathbb{D}$ such that, when $\varphi(z)\notin K$, $|u(z)|\leq\varepsilon$. Then we have

$$\|u.(f_n-f_m)\circ\varphi\|_\infty\leq\max\Big\{\|u\|_\infty.\sup_{\varphi(z)\in K}|(f_n-f_m)(z)|;2\varepsilon\Big\}$$

which is less than 2ε, when n,m are large enough, due to the uniform convergence on the compact set $\varphi(K)$. ■

Remark. Note that the preceding result implies that a composition operator C_φ on H^∞ is completely continuous if and only if it is weakly compact if and only if $\|\varphi\|_\infty<1$. Indeed, if $\|\varphi\|_\infty<1$, it is actually compact (and even nuclear) and if C_φ is completely continuous (resp. weakly compact) on H^∞ then its restriction to the disk algebra is as well. The result follows from the preceding theorem with $u=\mathbb{1}$.

We have the same results when the operators act on $A(\mathbb{D})$ (under the extra assumption that $\varphi\in A(\mathbb{D})$).

Corollary 1.2 *Let $u:\mathbb{D}\to\mathbb{C}$ be a bounded analytic map.*

1) Assume that $\varphi^{*-1}(\mathbb{T})$ has positive measure. Then $T_{u,\varphi}$ is weakly compact or completely continuous if and only if $u=0$.

2) In particular, if we assume that $M_u:A(\mathbb{D})\to H^\infty$ is weakly compact or completely continuous. Then $u=0$.

Proof: If $T_{u,\varphi}$ is weakly compact or completely continuous, it follows immediatly from the preceding theorem that $u^* = 0$ on a set of positive measure. As $u \in H^\infty$, we obtain that $u = 0$. ■

In this section, X denotes either $A(\mathbb{D})$ or H^∞.

In the sequel, we shall adapt our argument to compute essential norms. We generalize the proposition in the following way. This is also a generalization of the result of Zheng [19] in several directions.

We first have a majorization

Lemma 1.3 *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic map, $u : \mathbb{D} \to \mathbb{C}$ be a bounded analytic function.*

Then

$$\|T_{u,\varphi}\|_e \leq \inf\{2n_\varphi(u), \|u\|_\infty\}.$$

Proof: Obviously, $\|T_{u,\varphi}\|_e \leq \|T_{u,\varphi}\| = \|u\|_\infty$.

Fix $\varepsilon > 0$. There exists $r \in (0,1)$ such that $\sup\limits_{\substack{|\varphi(z)|\geq r\\ z\in\mathbb{D}}} |u(z)| \leq n_\varphi(u) + \varepsilon$. Denoting by F_N the Féjèr kernel, we consider the operator defined by

$$S(f)(z) = u(z).(F_N * f)\big(\varphi(z)\big) = u(z)\sum_{n=0}^{N}\left(1 - \frac{n}{N+1}\right)\hat{f}(n)\varphi^n(z),$$

where N is chosen large enough to verify $r^N \leq \varepsilon(1-r)$ and $\dfrac{1}{N+1}\displaystyle\sum_{n=1}^{N} nr^n \leq \varepsilon$.

S is a finite rank operator and the lemma is proved as soon as the following inequality holds

$$\|T_{u,\varphi} - S\| \leq \max(2n_\varphi(u) + 2\varepsilon, 2\varepsilon\|u\|_\infty).$$

Clearly, for every f in the unit ball of H^∞, $\|(T_{u,\varphi} - S)(f)\|$ is less than

$$\max\left\{\sup_{\substack{|\varphi(z)|\geq r\\ z\in\mathbb{D}}} |u(z)|.\Big|(f - F_N * f)\big(\varphi(z)\big)\Big|;\ \sup_{\substack{|\varphi(z)|\leq r\\ z\in\mathbb{D}}} |u(z)|.\Big|(f - F_N * f)\big(\varphi(z)\big)\Big|\right\}$$

We have

$$\sup_{\substack{|\varphi(z)|\geq r\\ z\in\mathbb{D}}} |u(z)|.\big|(f - F_N * f)\big(\varphi(z)\big)\big| \leq (n_\varphi(u) + \varepsilon)\sup_{w\in\mathbb{D}} |(f - F_N * f)(w)|$$

which is less than $2(n_\varphi(u) + \varepsilon)$, by the properties of the Féjèr kernel and the maximum modulus principle.

On the other hand, for every $z \in \mathbb{D}$ such that $|\varphi(z)| \leq r$, we have

$$|u(z)|.\big|(f - F_N * f)\big(\varphi(z)\big)\big| \leq \|u\|_\infty\left(\sum_{n=0}^{N}\frac{n}{N+1}\big|\hat{f}(n)\varphi^n(z)\big| + \sum_{n=N+1}^{\infty}\big|\hat{f}(n)\varphi^n(z)\big|\right)$$

so that

$$\sup_{\substack{|\varphi(z)|\leq r\\ z\in\mathbb{D}}} |u(z)|.\big|(f - F_N * f)\big(\varphi(z)\big)\big| \leq \|u\|_\infty\left(\sum_{n=0}^{N}\frac{n}{N+1}r^n + \frac{r^N}{1-r}\right) \leq 2\varepsilon\|u\|_\infty.$$

This gives the result. ■

Another kind of proof could be given. This will be done in the next section (see Lemma 2.4).

On the other hand , we have the lower estimate:

Lemma 1.4 *Let $u \in H^\infty$ and $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic map.*
We assume that $\mathcal{I} \subset \mathcal{W}(X, H^\infty) = \mathcal{DP}(X, H^\infty)$.
Then

$$n_\varphi(u) \le \|T_{u,\varphi}\|_{e,\mathcal{I}}.$$

Proof: The idea of the proof is a mix of the one of Theorem 1.1 and the one of Zheng [19]. We already know that $\|T_{u,\varphi}\|_{e,\mathcal{I}} = 0$ if and only if $T_{u,\varphi}$ is completely continuous if and only if $n_\varphi(u) = 0$ if and only if $T_{u,\varphi}$ is compact. We assume now that $T_{u,\varphi}$ is not completely continuous and this implies that $\|\varphi\|_\infty = 1$.

We choose a sequence $z_j \in \mathbb{D}$ such that $\varphi(z_j)$ converges to some $a \in \mathbb{T}$ and $|u(z_j)|$ converges to $n_\varphi(u)$.

We introduce the sequence of functions (where $n \ge 2$)

$$f_n(z) = \frac{n\bar{a}z - (n-1)}{n - (n-1)\bar{a}z},$$

which lies in the unit ball of the disk algebra.

Obviously, $f_n(z) \longrightarrow -1$ for every $z \in \overline{\mathbb{D}} \setminus \{a\}$ and $f_n(a) = 1$.

Now, let $S \in \mathcal{I}$. As the sequence $(f_n)_n$ is a weakly-Cauchy sequence, the sequence $\big(S(f_n)\big)_n$ is a norm Cauchy sequence, hence converging to some $\sigma \in H^\infty$. Observe that for every n,

$$\|(T_{u,\varphi} - S)(f_n)\|_\infty \ge \|T_{u,\varphi}(f_n) - \sigma\|_\infty - \|S(f_n) - \sigma\|_\infty$$

and we already know that $\|S(f_n) - \sigma\|_\infty \longrightarrow 0$.

For every $z \in \overline{\mathbb{D}} \setminus \{a\}$, we have $f_n(z) \longrightarrow -1$ so that for every $z \in \overline{\mathbb{D}} \setminus \{a\}$,

$$|u(z).f_n \circ \varphi(z) - \sigma(z)| \longrightarrow |u(z) + \sigma(z)|.$$

The proof splits in two cases:

Case 1. If $|u(w_0) + \sigma(w_0)| > n_\varphi(u)$ *for some* $w_0 \in \mathbb{D}$, then

$$\begin{aligned} \|T_{u,\varphi} - S\| &\ge \overline{\lim}\|(T_{u,\varphi} - S)(f_n)\|_\infty \ge \overline{\lim}|u(w_0).f_n \circ \varphi(w_0) - \sigma(w_0)| \\ &\ge |u(w_0) + \sigma(w_0)| \ge n_\varphi(u). \end{aligned}$$

Case 2. If not, then $\|u + \sigma\|_\infty \le n_\varphi(u)$ and it follows that for every $z \in \mathbb{D}$,

$$|u(z) - \sigma(z)| \ge 2|u(z)| - n_\varphi(u).$$

We have for every $n \ge 2$ and every integer j:

$$\begin{aligned} \|T_{u,\varphi} - S\| &\ge |u(z_j).f_n \circ \varphi(z_j) - \sigma(z_j)| - \|S(f_n) - \sigma\|_\infty \\ &\ge 2|u(z_j)| - n_\varphi(u) - |u(z_j)|.|f_n \circ \varphi(z_j) - 1| - \|S(f_n) - \sigma\|_\infty. \end{aligned}$$

Letting first j tend to infinity (and not forgetting that u is bounded on $\mathbb{D}$), we obtain for every integer $n \ge 2$

$$\|T_{u,\varphi} - S\| \ge n_\varphi(u) - \|S(f_n) - \sigma\|_\infty.$$

Then, letting n tend to infinity, we have $\|T_{u,\varphi} - S\| \ge n_\varphi(u)$. ■

Remark. We could have given another proof for weak compactness (avoiding the argument of the coincidence of $\mathcal{W}(X,H^\infty)$ and $\mathcal{DP}(X,H^\infty)$): this will be done below when we shall treat the general case (see Lemma 2.3).

The following theorem gives a generalization of previously known results on the subject.

Theorem 1.5 *Let $u\in H^\infty$ and $\varphi:\mathbb{D}\to\mathbb{D}$ be an analytic map. We assume that $\mathcal{K}(X,H^\infty)\subset\mathcal{I}\subset\mathcal{W}(X,H^\infty)=\mathcal{DP}(X,H^\infty)$.*

Then

$$\|T_{u,\varphi}\|_{e,\mathcal{I}}\approx n_\varphi(u).$$

More precisely

$$n_\varphi(u)\le\|T_{u,\varphi}\|_{e,\mathcal{I}}\le\inf\{2n_\varphi(u),\|u\|_\infty\}.$$

As a particular case, when $n_\varphi(u)=\|u\|_\infty$, the following equality holds:

$$\|T_{u,\varphi}\|_{e,\mathcal{I}}=\|T_{u,\varphi}\|_e=\|u\|_\infty.$$

Proof: We obviously have $\|T_{u,\varphi}\|_{e,\mathcal{I}}\le\|T_{u,\varphi}\|_e$. The result follows from the two preceding lemmas. ■

The following immediatly follows

Corollary 1.6 *Let $u:\mathbb{D}\to\mathbb{C}$ be a bounded analytic map. Then*

$$\|M_u\|_{e,\mathcal{I}}=\|M_u\|_e=\|u\|_\infty.$$

We are able to obtain the exact value of the essential norms of the operators $T_{u,\varphi}$ when φ has some contacts with $\mathbb{T}$ on thin sets, for instance on a finite set. More generally, we define the property $\mathcal{T}$ for $\varphi:\mathbb{D}\to\mathbb{D}$ (analytic) by

There exists a compact set $K\subset\mathbb{T}$, with Haar measure 0, such that:
for every sequence (z_n) in $\mathbb{D}$ converging to $k\in\mathbb{D}$, with $|\varphi(z_n)|\to1$, we have $k\in K$.

Clearly, if $\varphi\in A(\mathbb{D})$, this property means that $\varphi^{-1}(\mathbb{T})$ has Haar measure 0.

Proposition 1.7 *Let $u\in A(\mathbb{D})$ and $\varphi:\mathbb{D}\to\mathbb{D}$ be an analytic map with property $\mathcal{T}$. We assume that $\mathcal{K}(X,H^\infty)\subset\mathcal{I}\subset\mathcal{W}(X,H^\infty)=\mathcal{DP}(X,H^\infty)$.*

Then

$$\|T_{u,\varphi}\|_{e,\mathcal{I}}=\|T_{u,\varphi}\|_e=n_\varphi(u).$$

Actually, the proof shows that we could assume only that u is bounded and $u_{|K}$ is continuous.

Proof: By the Rudin-Carleson Theorem, there exists some $v\in A(\mathbb{D})$, with $v=u$ on K and $\|v\|_\infty\le\|u_{|K}\|_\infty$.

We claim that $T_{u-v,\varphi}$ is compact since $n_\varphi(u-v)=0$. Indeed, we have $n_\varphi(u-v)\le\sup_K|u-v|$ by property $\mathcal{T}$.

Obviously, $\|T_{u,\varphi}\|_e\le\|T_{u,\varphi}-T_{u-v,\varphi}\|=\|T_{v,\varphi}\|\le\|v\|_\infty\le\|u_{|K}\|_\infty=n_\varphi(u)$. ■

2 The several variable case.

We are going to fix a general frame: B shall denote in this section the open unit ball of a complex Banach space $(E, \|.\|)$. A function $f : B \to \mathbb{C}$ is analytic if it is Fréchet differentiable. The space $H^\infty(B)$ is then the space of bounded analytic functions on B (see [9] to know more on the subject). The space $A(B)$ the space of uniformly continuous analytic functions on B. These two spaces are equipped with the uniform norm

$$\|f\|_\infty = \sup_{z\in B} |f(z)|.$$

The case $E = \mathbb{C}$ and $B = \mathbb{D}$ corresponds clearly to the classical case. When $d \geq 2$, we have the two following special cases, which we are particularly interested in:

- When $\mathbb{C}^d$ is equipped with the sup-norm $\|(z_1, \dots, z_d)\|_\infty = \max\limits_{1\leq j\leq d} |z_j|$, the framework corresponds to the polydisk algebra: $B = \mathbb{D}^d$.

- When $\mathbb{C}^d$ is equipped with the hermitian norm $\|(z_1, \dots, z_d)\|_2^2 = \sum\limits_{j=1}^{d} |z_j|^2$, the framework corresponds to the polyball algebra: $B = \mathbb{B}^d$.

Let φ be an analytic function from B into itself and C_φ the associated composition operator. In the sequel,

$$\|\varphi\|_\infty = \sup_{z\in B} \|\varphi(z)\|.$$

We also consider $u \in H^\infty(B)$ and we are going to study the operator $T_{u,\varphi}$. We shall denote by X either $A(B)$ or $H^\infty(B)$.

The results are essentially the same than the one obtained in the one variable case. The proofs mainly use the same ideas and tools. Nevertheless, we first need the following specific lemma.

Lemma 2.1 *Let ξ in the unit ball of the dual of E.*
The sequence of functions

$$z \in B \longmapsto f_n(z) = \frac{n\xi(z) - (n-1)}{n - (n-1)\xi(z)}$$

is a weak Cauchy sequence in the space $A(B)$.

Proof: First, (f_n) clearly belongs to the unit ball of $A(B)$. Actually, the key point is that $f_n = F_n \circ \xi$, where $F_n(t) = \dfrac{nt - (n-1)}{n - (n-1)t}$ is a weak Cauchy sequence in the space of continuous function on $\overline{\mathbb{D}}$. Obviously, $F_n(t) \longrightarrow -1$ for every $t \in \overline{\mathbb{D}}$, with $t \neq 1$. We have $f_n(1) = 1$.

Let $\nu \in A(B)^*$. By the Hahn-Banach theorem, we obtain $\tilde{\nu}$ belonging to the dual of $C(\overline{B})$, the space of continuous functions on $\overline{B}$ (the norm closure of B). We can define a linear continuous functional χ on $C(\overline{\mathbb{D}})$ in the following way

$$h \in C(\overline{\mathbb{D}}) \longmapsto \tilde{\nu}(h \circ \xi)$$

The classical Riesz representation Theorem gives us a Borel measure μ on $\overline{\mathbb{D}}$ such that for every $h \in C(\overline{\mathbb{D}})$, $\chi(h) = \int_{\overline{\mathbb{D}}} h d\mu$. But, we have by the Lebesgue domination theorem,

$$\nu(f_n) = \chi(F_n) \longrightarrow \mu\big(\{1\}\big) - \mu\big(\overline{\mathbb{D}} \setminus \{1\}\big).$$

This implies that the sequence $\big(f_n\big)$ is a weak Cauchy sequence in the space $A(B)$. ■

The following result is a consequence of Montel's Theorem, similar to Th.1.1 ($3 \Rightarrow 4$).

Lemma 2.2 *[1] Let $\varphi : B \to B$ be an analytic map. We assume that $\|\varphi\|_\infty < 1$ and $\varphi(B)$ relatively compact. Then C_φ is compact.*

Proof: See [1],Prop. 3. ■

We have the following lemma, similar to Lemma 1.4.

Lemma 2.3 *Let $u \in H^\infty(B)$ and $\varphi : B \to B$ be analytic with $\|\varphi\|_\infty = 1$ and $\varphi(B)$ relatively weakly compact. We assume that $\mathcal{I} \subset \mathcal{W}(X, H^\infty(B)) + \mathcal{DP}(X, H^\infty(B))$.*
Then

$$n_\varphi(u) \le \|T_{u,\varphi}\|_{e,\mathcal{I}}.$$

When E is reflexive (e.g. finite dimensional), the relative weak compactness assumption on $\varphi(B)$ is automatically fullfiled.

Proof: First, we begin with some preliminary remarks. There exists $z_j \in B$ such that $\|\varphi(z_j)\|$ converges to 1. Up to an extraction, we may suppose that $\varphi(z_j)$ is weakly converging to some $a \in \overline{B}$. Actually, $\|a\| = 1$. We choose ξ in the unit sphere of the dual of E such that $\xi(a) = \|a\| = 1$.

We introduce the sequence of functions $z \in B \longmapsto f_n(z) = \dfrac{n\xi(z) - (n-1)}{n - (n-1)\xi(z)}$, which clearly lies in the unit ball of $A(B)$.

Let $S \in \mathcal{I}$. The assumption gives $S = D + W$, where $D \in \mathcal{DP}(X, H^\infty(B))$ and $W \in \mathcal{W}(X, H^\infty(B))$.

By Lemma 2.1, the sequence $(f_n)_n$ is a weak-Cauchy sequence, the sequence $\big(D(f_n)\big)_n$ is a norm Cauchy sequence, hence converges to some $\Delta \in H^\infty(B)$. On the other hand, a subsequence of $W(f_n)$ is weakly convergent to some $w \in H^\infty(B)$, so by the Mazur Theorem, we can find some $c_k \ge 0$ with $\displaystyle\sum_{k \in I_m} c_k = 1$, where $I_m \subset \mathbb{N}$; and $\displaystyle\sum_{k \in I_m} c_k W(f_k) \to w$. Moreover, we can assume that $\sup I_m < \inf I_{m+1}$.

Writing $\tilde{f}_m = \displaystyle\sum_{k \in I_m} c_k f_k$, we have: for every $z \in B$, $\tilde{f}_m(z) \to -1$ and for every m, $\tilde{f}_m\big(\varphi(z_j)\big) \to 1$. It is clear that $\big(D(\tilde{f}_m)\big)_m$ is norm convergent to Δ, so $\big(S(\tilde{f}_m)\big)_m$ is norm convergent to $\sigma = \Delta + w$.

The argument now follows the lines of the proof of Lemma 1.4 and we obtain

$$\|T_{u,\varphi} - S\| \ge n_\varphi(u).$$

■

For the upper estimate, we have a result similar to Lemma 1.3. We present here an alternative argument.

Lemma 2.4 *Let $\varphi : B \to B$ be analytic with $\varphi(B)$ relatively compact and $u : \mathbb{D} \to \mathbb{C}$ be a bounded analytic function. Then*

$$\|T_{u,\varphi}\|_e \le \inf\{2n_\varphi(u), \|u\|_\infty\}.$$

Proof: Fix $\varepsilon > 0$. There exists $r \in (0,1)$ such that

$$\sup_{\substack{\|\varphi(z)\|\geq r\\ z\in B}} |u(z)| \leq n_\varphi(u) + \varepsilon.$$

We consider the operator defined by

$$S(f)(z) = u(z).f\big(\rho\varphi(z)\big),$$

where ρ is chosen in $(0,1)$, close enough to 1 to verify $\displaystyle\sum_{n\geq 0}\big(1-\rho^n\big)r^n \leq \varepsilon$.

By Lemma 2.2, S is a compact operator since $\|\rho\varphi\|_\infty \leq \rho < 1$ and $\varphi(B)$ is relatively compact.

For every f in the unit ball of X and every $z \in B$, we have the Taylor expansion

$$f(z) = \sum_{n\geq 0} \frac{1}{n!} d^n f(0).(z)^{(n)}$$

where $d^n f(0)$ denotes the n^{th} differential in the point 0, and $(z)^{(n)} = (z, \ldots, z)$.

Hence, $\big(T_{u,\varphi} - S\big)(f)(z) = u(z)\displaystyle\sum_{n\geq 0}\frac{1}{n!}\big(1-\rho^n\big)d^n f(0).\big(\varphi(z)\big)^{(n)}$.

Since $\dfrac{1}{n!}\big\|d^n f(0)\big\| \leq \|f\|_\infty \leq 1$, we obtain, when $\|\varphi(z)\| \leq r$,

$$\big\|\big(T_{u,\varphi} - S\big)(f)(z)\big\| \leq \|u\|_\infty \sum_{n\geq 0}\big(1-\rho^n\big)\|\varphi(z)\|^n \leq \|u\|_\infty.\varepsilon.$$

On the other hand, when $\|\varphi(z)\| \geq r$, we have

$$\big\|\big(T_{u,\varphi} - S\big)(f)(z)\big\| \leq (n_\varphi(u)+\varepsilon)\Big(\big|f\big(\varphi(z)\big)\big| + \big|f\big(\rho\varphi(z)\big)\big|\Big) \leq 2(n_\varphi(u)+\varepsilon).$$

Finally,

$$\big\|T_{u,\varphi} - S\big\| \leq \max\big\{\varepsilon\|u\|_\infty; 2(n_\varphi(u)+\varepsilon)\big\}.$$

As $\varepsilon > 0$ is arbitrary, we conclude $\|T_{u,\varphi}\|_e \leq 2n_\varphi(u)$. This gives the result. ■

The following theorem is the main result of this section.

Theorem 2.5 *Let $u \in H^\infty(B)$ and $\varphi : B \to B$ be analytic. We assume that $\varphi(B)$ is relatively compact and $\mathcal{K}(X, H^\infty(B)) \subset \mathcal{I} \subset \mathcal{W}(X, H^\infty(B)) + \mathcal{DP}(X, H^\infty(B))$.*

Then

$$\|T_{u,\varphi}\|_{e,\mathcal{I}} \approx n_\varphi(u).$$

More precisely

$$n_\varphi(u) \leq \|T_{u,\varphi}\|_{e,\mathcal{I}} \leq \inf\{2n_\varphi(u), \|u\|_\infty\}.$$

As a particular case, when $n_\varphi(u) = \|u\|_\infty$, the following equality holds:

$$\|T_{u,\varphi}\|_{e,\mathcal{I}} = \|T_{u,\varphi}\|_e = \|u\|_\infty.$$

Of course, when E is finite dimensional, the compactness assumption can be forgotten.
Proof: This is an immediate consequence of the preceding lemmas. ■

We specify two particular cases.

Corollary 2.6 *Let $u \in H^\infty(B)$ and $\varphi : B \to B$ be analytic with $\varphi(B)$ relatively compact. We assume that $\mathcal{K}(X, H^\infty(B)) \subset \mathcal{I} \subset \mathcal{W}(X, H^\infty(B)) + \mathcal{DP}(X, H^\infty(B))$.*

1) $\|M_u\|_{e,\mathcal{I}} = \|M_u\|_e = \|u\|_\infty$.

2) $\|C_\varphi\|_{e,\mathcal{I}} = 1$ *if* $\|\varphi\|_\infty = 1$ *and* $\|C_\varphi\|_{e,\mathcal{I}} = 0$ *if* $\|\varphi\|_\infty < 1$.

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